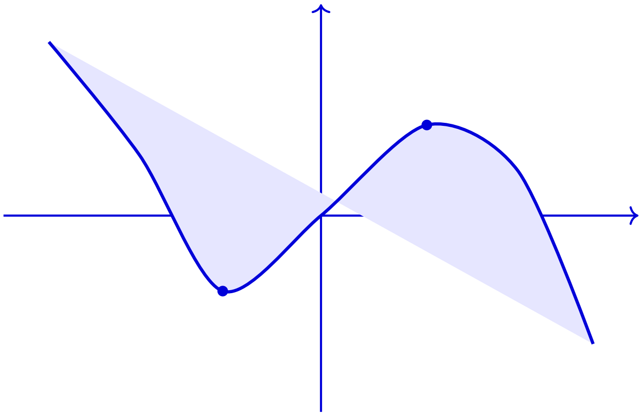
Summit MATH 252: Differential Equations

Summit fully illustrated textbook edition



Original Summit-authored instructional text generated from the live course runtime, bibliography layer, and assessment structure.

March 22, 2026

@@TOKEN_0@@ Summit first edition draft @@TOKEN_1@@ college @@TOKEN_2@@ 4 @@TOKEN_3@@ 14 weeks @@TOKEN_4@@ 9.6 hours/week

Originality note

This textbook is a Summit-authored instructional text. It is informed by the course bibliography in @@TOKEN_0@@ and by open academic references used elsewhere in Summit, but it does not copy or restate any single commercial textbook.

How this textbook was built

This book was generated from the live Summit course runtime for Differential Equations: the syllabus, lesson sequence, reading chapters, guided practice, homework sets, quizzes, mastery exam, and workload standard. The design goal is to give a student a usable, course-complete book while preserving original Summit wording and sequencing.

An original Summit course in ordinary differential equations focused on first-order models, second-order systems, Laplace transforms, and matrix-based system behavior in engineering contexts.

Mathematics chapters should move from concept to representation to fluent execution. Students should always know what the symbols mean before they try to manipulate them.

This volume is structured as a teaching book rather than a bare note pack. Every chapter contains explanation, worked examples, guided practice, chapter homework, and a rear answer key so the student can study independently and still get disciplined feedback.

Course use guide

- Read one chapter at a time in sequence; each chapter is aligned to a live lesson block in the course workspace.
- Rebuild the worked examples before attempting the graded homework or quiz material.
- Keep a scratch notebook beside the text and write down assumptions, diagrams, and the points where you usually get stuck.
- Use the course tutor, guided practice, and homework only after you can explain the chapter in your own words.

Contents

Course map

- 4 live lesson chapters
- 2 graded homework checkpoints
- 2 timed quizzes
- 1 cumulative mastery exam
- 6 declared course outcomes

Prerequisite and readiness position

Course prerequisites: calculus-ii, calculus-iii. Readiness clearances: calc-ii-credit, multivariable-ready, matrix-ready.

Summit Differential Equations assumes strong integration fluency, comfort with multivariable ideas, and enough matrix reasoning to work with systems and eigenvalue-driven models.

Semester workload standard

Summit models this course as @@TOKEN_0@@ across a 14-week term plus final assessment window. The expected distribution is:

- Contact-equivalent instruction: 42 hours
- Reading: 14 hours
- Practice and problem solving: 44 hours
- Homework: 20 hours
- Lab, design, and reporting: 0 hours
- Exam preparation: 15 hours

Expected volume:

- 130-160 ODE, Laplace, systems, and stability problems emphasizing method selection and interpretation.
- 8-10 graded homework sets totaling 32-42 multi-step problems with full written solutions.
- No standalone lab block; explanation and correction work is embedded inside problem solving and exam review.

Reference basis

Primary synthesis anchors from the bibliography for this course (50 listed references total):

1. Differential Equations and Linear Algebra
2. Elementary Differential Equations and Boundary Value Problems
3. Differential Equations with Boundary-Value Problems
4. Notes on Diffy Qs
5. Fundamentals of Differential Equations
6. Engineering Differential Equations
7. Differential Equations for Engineers
8. Differential Equations

Chapter 1

Chapter 1 First-order equations and modeling

Chapter purpose

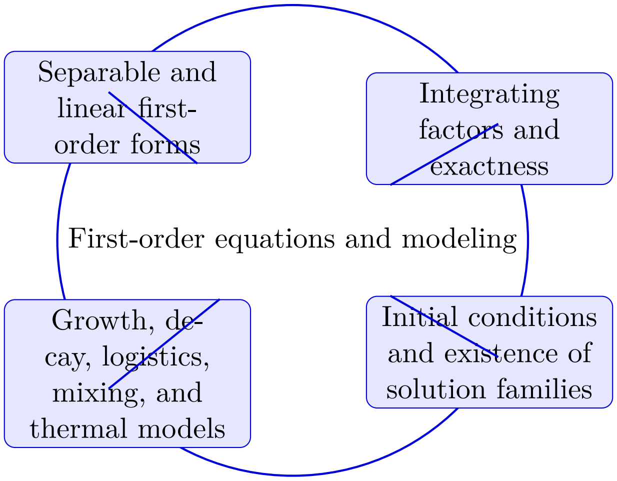
Students begin by learning that the equation type determines the solution method. The lesson covers separable equations, linear first-order equations with integrating factors, exact equations, equilibrium analysis, and classical models for growth, decay, mixing, and Newtonian cooling. Interpretation is central: every solution should be checked against physical context and units.

This chapter sits at the opening of Differential Equations. It develops Separable and linear first-order forms, Integrating factors and exactness, Initial conditions and existence of solution families, and Growth, decay, logistics, mixing, and thermal models so that the student can move from explanation to execution without losing the thread of the course.

The central habit in this chapter is to move across words, graphs, formulas, and worked algebra without losing meaning. A correct answer is not enough on its own; the student should be able to explain why the setup is valid and how the result fits the larger mathematical structure of the course.

Core ideas

- Separable and linear first-order forms
- Integrating factors and exactness
- Initial conditions and existence of solution families
- Growth, decay, logistics, mixing, and thermal models



How to think through this chapter

Problem solving in this family starts with naming the structure of the task. Students should ask which theorem, definition, or representation controls the problem before choosing a computational path. Once the structure is clear, algebraic execution should be clean, annotated, and checked against the expected behavior of the function or model.

When working this chapter, keep the following question active: @@TOKEN_0@@ A good student answer should connect setup, assumptions, and conclusion instead of only chasing a final number or sentence.

Differential equations begin when a rate law becomes the main object of study. Instead of solving for a quantity directly, the course starts with how the quantity changes and asks what function could obey that rule.

A differential equation is a law of motion or change

This shift is one of the most important in applied mathematics. Many real systems do not hand us a clean formula for the quantity of interest. They tell us how that quantity evolves. Population growth, cooling, charging, and mixing problems all arrive that way.

Students should notice that the unknown in a differential equation is not just a number. It is an entire function. That changes the level of thinking required.

Methods succeed because structure matters

Separable equations, linear equations, and integrating factors are not unrelated tricks. They are responses to structural features of the equation. If the variables can be separated, do it. If the equation is linear, the integrating factor rescales it into a more cooperative form.

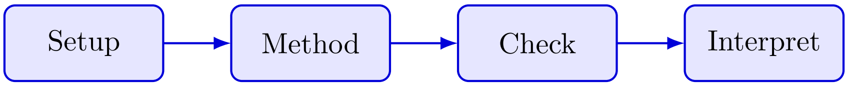
This chapter rewards method recognition more than raw algebra speed. The smartest first step is classification.

Initial conditions turn families into models

Most early solutions produce a family of curves. The constant of integration records that many different histories can obey the same rate law. An initial condition picks the one curve matching the actual system state.

That is a powerful modeling lesson: governing laws and present state work together. One without the other is incomplete.

Worked example



@@TOKEN_0@@ Solve y' = 3y with y(0) = 2.

1. This is separable: dy / y = 3 dx.
2. Integrate to get ln|y| = 3x + C.
3. Exponentiate: y = Ce^(3x).
4. Apply y(0) = 2 to obtain y = 2e^(3x).

Read this example twice: once for the flow of ideas and once for the technical structure of the solution.

Worked-through guided example

@@TOKEN_0@@ Solve dy/dx = 3x^2 with y(0) = 4.

1. Integrate dy = 3x^2 dx.
2. Write the general solution with a constant of integration.
3. Use y(0) = 4 to determine the constant.

Integrating gives y = x^3 + C. The initial condition implies C = 4, so y = x^3 + 4.

Instructor commentary

Students should annotate this chapter for structure, not just facts. Mark where the argument changes direction, where the method requires a hidden assumption, and where the conclusion becomes more general than the worked example. If the chapter feels easy while you are reading it but difficult when you close the page, you have not yet converted recognition into mastery.

The most effective study pattern is read, annotate, rebuild the worked example without looking, and then solve several short-to-long problems in one sitting so the idea becomes automatic.

Practice while you read

Practice Set: First-order equations

Classify separable and linear equations correctly before starting the algebra.

@@TOKEN_0@@ Solve dy/dx = 3x^2 with y(0) = 4.

- Hint: This is already separated, so integrate both sides directly before using the initial condition.
- Step 1: Integrate dy = 3x^2 dx.
- Step 2: Write the general solution with a constant of integration.
- Step 3: Use y(0) = 4 to determine the constant.
- Checkpoint: y = x^3 + 4

@@TOKEN_0@@ Solve y’ + y = 0.

- Hint: A linear first-order equation with zero forcing can be separated or solved by the linear-equation method.
- Step 1: Rewrite as dy/dx = -y.
- Step 2: Separate variables to get dy/y = -dx.
- Step 3: Integrate both sides and solve for y.
- Checkpoint: y = Ce^(-x)

Chapter homework

@@TOKEN_0@@ Method recognition, integrating factors, characteristic roots, and forcing.

1. Solve y’ + 2y = e^x with y(0) = 1.
2. A tank initially holds 100 liters of pure water. Brine with concentration 0.2 kg/L flows in at 3 L/min and the well-mixed solution exits at 3 L/min. Write and solve the IVP for salt mass.

3. Solve y” + 4y = 0 with y(0) = 2 and y’(0) = -1.
4. Find a particular solution to y” - y’ - 6y = 5e^(2x).

Answers for these homework problems appear in the back-of-book answer key.

Chapter summary and study notes

- Identify the equation form before attempting algebra.
- Use an integrating factor without losing constants or domains.
- Check equilibrium solutions and long-term behavior.

Study tips

- Classify the equation before trying to solve it.
- Carry the differential notation carefully when separating variables.
- Use the initial condition only after the general solution form is clear.

Common traps

- Trying an integrating factor on an equation that is easier to separate.
- Losing absolute values in logarithmic integration steps.
- Applying an initial condition to an algebraically incomplete solution.

Family-level errors to watch for

- Starting algebra before identifying the governing definition or theorem.
- Dropping notation, units, or sign conventions in the middle of a calculation.
- Treating a symbolic answer as finished without interpreting what it means.

Chapter 2

Chapter 2 Second-order linear equations

Chapter purpose

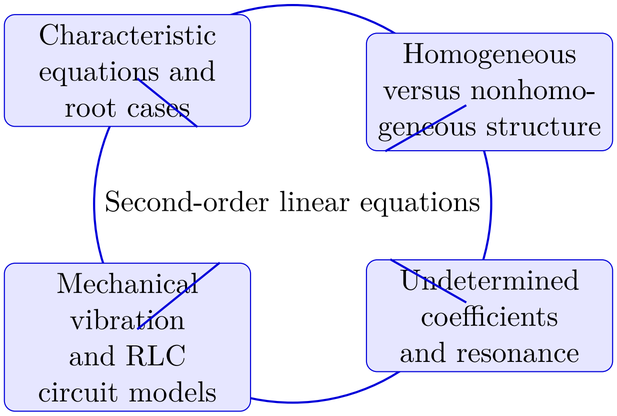
The course then shifts to second-order systems, especially the constant-coefficient equations used in mechanics and circuits. Students solve homogeneous equations by characteristic roots, then build particular solutions for forcing terms using undetermined coefficients or variation of parameters. The lesson pays special attention to repeated roots, complex roots, resonance, and physical interpretation.

This chapter sits in the middle of Differential Equations. It develops Characteristic equations and root cases, Homogeneous versus nonhomogeneous structure, Undetermined coefficients and resonance, and Mechanical vibration and RLC circuit models so that the student can move from explanation to execution without losing the thread of the course.

The central habit in this chapter is to move across words, graphs, formulas, and worked algebra without losing meaning. A correct answer is not enough on its own; the student should be able to explain why the setup is valid and how the result fits the larger mathematical structure of the course.

Core ideas

- Characteristic equations and root cases
- Homogeneous versus nonhomogeneous structure
- Undetermined coefficients and resonance
- Mechanical vibration and RLC circuit models



How to think through this chapter

Problem solving in this family starts with naming the structure of the task. Students should ask which theorem, definition, or representation controls the problem before choosing a computational path. Once the structure is clear, algebraic execution should be clean, annotated, and checked against the expected behavior of the function or model.

When working this chapter, keep the following question active: @@TOKEN_0@@ A good student answer should connect setup, assumptions, and conclusion instead of only chasing a final number or sentence.

Second-order equations matter because many physical systems store both position and momentum-like behavior. Springs, circuits, and structural vibrations need a model rich enough to remember more than one layer of change.

Second order means the system has memory

A first-order model often reacts only to the current state. A second-order model remembers both state and how that state is moving. That extra layer is why oscillation appears so naturally in this chapter.

Students should connect this to mechanics early. If position is changing and velocity is also changing, one derivative is not enough to capture the full story.

Homogeneous solutions reveal natural behavior

The homogeneous equation strips away forcing and shows what the system would do on its own. Characteristic roots then reveal whether the behavior decays, grows, or oscillates. This is one of the most useful diagnostic tools in applied mathematics.

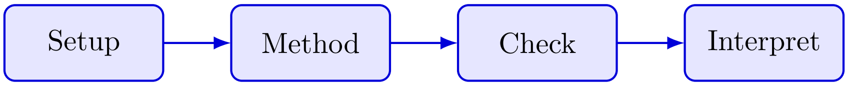
A good habit is to interpret the roots before writing the full symbolic solution. Real distinct roots, repeated roots, and complex roots each tell a different physical story.

Particular solutions show how outside forcing enters

Once forcing is added, the solution splits naturally into intrinsic behavior plus response to the input. That split is conceptually valuable. It separates what belongs to the system from what belongs to the environment acting on it.

This viewpoint helps students understand resonance and long-term behavior later, because the forcing is no longer just another term in the equation. It is a driver interacting with the system dynamics.

Worked example



@@TOKEN_0@@ Solve y” - 3y’ + 2y = 0.

1. Form the characteristic equation r^2 - 3r + 2 = 0.
2. Factor to get (r - 1)(r - 2) = 0, so roots are 1 and 2.
3. Distinct real roots give y = C1 e^x + C2 e^(2x).

Read this example twice: once for the flow of ideas and once for the technical structure of the solution.

Worked-through guided example

@@TOKEN_0@@ Solve y” - 5y’ + 6y = 0.

1. Build the characteristic equation r^2 - 5r + 6 = 0.
2. Factor it to find the roots.
3. Write the homogeneous solution using those roots.

The characteristic equation factors as (r - 2)(r - 3) = 0, giving roots 2 and 3. Therefore y = C1e^(2x) + C2e^(3x).

Instructor commentary

Students should annotate this chapter for structure, not just facts. Mark where the argument changes direction, where the method requires a hidden assumption, and where the conclusion becomes more general than the worked example. If the chapter feels easy while you are reading it but difficult when you close the page, you have not yet converted recognition into mastery.

The most effective study pattern is read, annotate, rebuild the worked example without looking, and then solve several short-to-long problems in one sitting so the idea becomes automatic.

Practice while you read

Practice Set: Second-order linear equations

Read characteristic roots and connect them to the behavior of the solution family.

@@TOKEN_0@@ Solve y” - 5y’ + 6y = 0.

- Hint: Use the characteristic polynomial and factor it before writing the general solution.
- Step 1: Build the characteristic equation r^2 - 5r + 6 = 0.
- Step 2: Factor it to find the roots.
- Step 3: Write the homogeneous solution using those roots.
- Checkpoint: y = C1e^(2x) + C2e^(3x)

@@TOKEN_0@@ Solve y” + 4y = 0.

- Hint: The characteristic roots are complex, so write the real-valued solution with sine and cosine.
- Step 1: Set up the characteristic equation r^2 + 4 = 0.
- Step 2: Solve for the complex roots.
- Step 3: Translate those roots into the sine-cosine solution form.
- Checkpoint: y = C1 cos(2x) + C2 sin(2x)

Chapter homework

@@TOKEN_0@@ Method recognition, integrating factors, characteristic roots, and forcing.

1. Solve y’ + 2y = e^x with y(0) = 1.
2. A tank initially holds 100 liters of pure water. Brine with concentration 0.2 kg/L flows in at 3 L/min and the well-mixed solution exits at 3 L/min. Write and solve the IVP for salt mass.
3. Solve y” + 4y = 0 with y(0) = 2 and y’(0) = -1.
4. Find a particular solution to y” - y’ - 6y = 5e^(2x).

Answers for these homework problems appear in the back-of-book answer key.

Chapter summary and study notes

- Translate root structure into the correct basis of solutions.
- Detect resonance before guessing a particular solution.
- Interpret damping, forcing, and oscillation in model language.

Study tips

- Interpret the characteristic roots before simplifying the full answer.
- Keep homogeneous and particular parts mentally separate.
- Check whether the forcing term overlaps with the homogeneous family before guessing a particular solution form.

Common traps

- Forgetting the extra x factor when a repeated root appears.
- Using a particular-solution guess that duplicates part of the homogeneous solution.
- Missing the physical meaning of damping, oscillation, or growth hidden in the roots.

Family-level errors to watch for

- Starting algebra before identifying the governing definition or theorem.
- Dropping notation, units, or sign conventions in the middle of a calculation.
- Treating a symbolic answer as finished without interpreting what it means.

Chapter 3

Chapter 3 Laplace transforms and discontinuous forcing

Chapter purpose

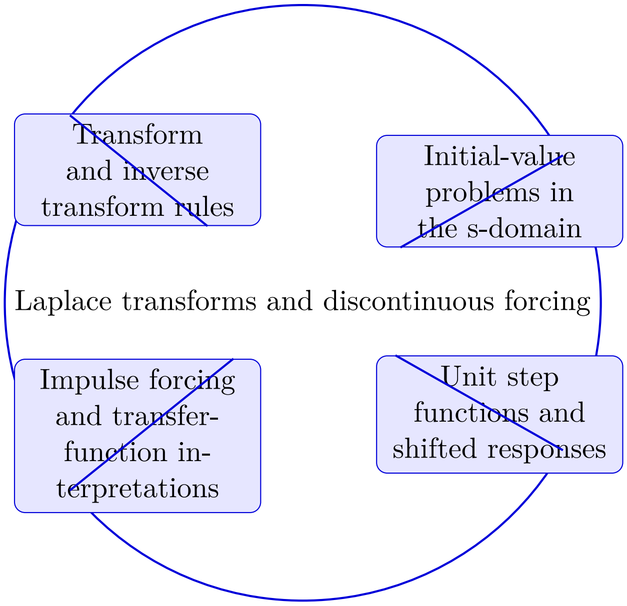
Laplace transforms let students convert differential equations into algebraic equations in the transform domain. This lesson introduces transform tables, derivative rules, inverse transforms, step functions, impulses, and transfer-function style thinking. Students see why transforms are powerful whenever forcing is piecewise, impulsive, or awkward for purely time-domain methods.

This chapter sits in the middle of Differential Equations. It develops Transform and inverse transform rules, Initial-value problems in the s-domain, Unit step functions and shifted responses, and Impulse forcing and transfer-function interpretations so that the student can move from explanation to execution without losing the thread of the course.

The central habit in this chapter is to move across words, graphs, formulas, and worked algebra without losing meaning. A correct answer is not enough on its own; the student should be able to explain why the setup is valid and how the result fits the larger mathematical structure of the course.

Core ideas

- Transform and inverse transform rules
- Initial-value problems in the s-domain
- Unit step functions and shifted responses
- Impulse forcing and transfer-function interpretations



How to think through this chapter

Problem solving in this family starts with naming the structure of the task. Students should ask which theorem, definition, or representation controls the problem before choosing a computational path. Once the structure is clear, algebraic execution should be clean, annotated, and checked against the expected behavior of the function or model.

When working this chapter, keep the following question active: @@TOKEN_0@@ A good student answer should connect setup, assumptions, and conclusion instead of only chasing a final number or sentence.

The Laplace transform gives students a new strategy: stop solving the differential equation directly and move the problem into an algebraic domain where differentiation becomes multiplication by s plus initial-data terms.

Transforms change the battlefield

The transform is useful because some problems are hard in time but easier in the transform domain. Initial conditions enter neatly, discontinuous forcing can be represented compactly, and algebra often replaces repeated differentiation.

Students should see this as a strategic change of representation, not as a mysterious new kind of magic. The problem stays the same. The language changes.

Tables matter, but understanding the pattern matters more

Transform tables are unavoidable, yet they are not the heart of the chapter. The heart is understanding how elementary functions, shifting, and convolution reshape the problem. Once students see the pattern, the table becomes a tool rather than a wall of facts.

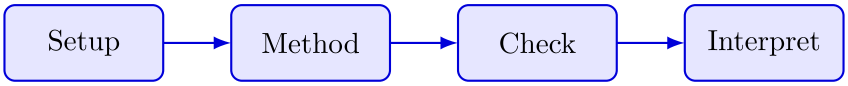
A good practical habit is to annotate each transformed term with its source: derivative rule, initial condition, shift, or lookup. That makes inversion far less error-prone.

Piecewise and impulse inputs are why this method earns its place

Laplace methods shine when the forcing function is not smooth. Switches, sudden inputs, and delayed signals appear naturally in engineering systems. The transform handles these without forcing the student to restart the problem on every interval by hand.

That is why this unit often feels more engineering-centered than earlier chapters. It models control over signals and timing directly.

Worked example



@@TOKEN_0@@ Find the Laplace transform of y’ if y(0) = y0.

1. The derivative rule is L{y’} = sY(s) - y(0).
2. Substitute the initial condition to get sY(s) - y0.
3. This formula is what converts an IVP into an algebra problem in s.

Read this example twice: once for the flow of ideas and once for the technical structure of the solution.

Worked-through guided example

@@TOKEN_0@@ Find L{t^2}.

1. Identify n = 2.
2. Substitute into the factorial-over-power formula.
3. Simplify the expression.

With n = 2, the rule gives 2! / s^3 = 2 / s^3.

Instructor commentary

Students should annotate this chapter for structure, not just facts. Mark where the argument changes direction, where the method requires a hidden assumption, and where the conclusion becomes more general than the worked example. If the chapter feels easy while you are reading it but difficult when you close the page, you have not yet converted recognition into mastery.

The most effective study pattern is read, annotate, rebuild the worked example without looking, and then solve several short-to-long problems in one sitting so the idea becomes automatic.

Practice while you read

Practice Set: Laplace transforms

Use transforms to move the problem into algebra and keep the initial-condition terms visible.

@@TOKEN_0@@ Find L{t^2}.

- Hint: Use the standard transform rule L{t^n} = n! / s^(n+1).
- Step 1: Identify n = 2.
- Step 2: Substitute into the factorial-over-power formula.
- Step 3: Simplify the expression.
- Checkpoint: L{t^2} = 2 / s^3

@@TOKEN_0@@ Solve y' + y = 0 with y(0) = 3 using a Laplace transform.

- Hint: Transform the derivative carefully so the initial value appears explicitly.
- Step 1: Take the Laplace transform of both sides to get sY(s) - 3 + Y(s) = 0.
- Step 2: Solve algebraically for Y(s).
- Step 3: Invert the resulting transform.
- Checkpoint: y = 3e^(-t)

Chapter homework

@@TOKEN_0@@ Transform techniques, step forcing, and qualitative behavior of linear systems.

1. Use Laplace transforms to solve y" + y = 0 with y(0) = 0 and y'(0) = 1.
2. Write the forcing function that turns on a constant load of 6 at t = 3 using the unit step function.

3. For the system x' = 3x + y, y' = x + 3y, find the eigenvalues.
4. Classify the equilibrium at the origin for x' = -2x, y' = -5y.

Answers for these homework problems appear in the back-of-book answer key.

Chapter summary and study notes

- Carry initial conditions through transform formulas carefully.
- Use partial fractions in the transform domain cleanly.
- Recognize when a step or impulse description is more natural than a piecewise time formula.

Study tips

- Write transformed derivatives term by term so the initial-condition contributions are visible.
- Use partial fractions only after the transformed equation is fully simplified.
- Check whether a time shift or step function is the cleanest way to represent the forcing.

Common traps

- Dropping initial-condition terms during the transform of derivatives.
- Inverting a transformed expression without first breaking it into recognizable pieces.
- Treating the transform table as memorization only instead of as pattern recognition.

Family-level errors to watch for

- Starting algebra before identifying the governing definition or theorem.
- Dropping notation, units, or sign conventions in the middle of a calculation.
- Treating a symbolic answer as finished without interpreting what it means.

Chapter 4

Chapter 4 Systems, phase portraits, and advanced modeling

Chapter purpose

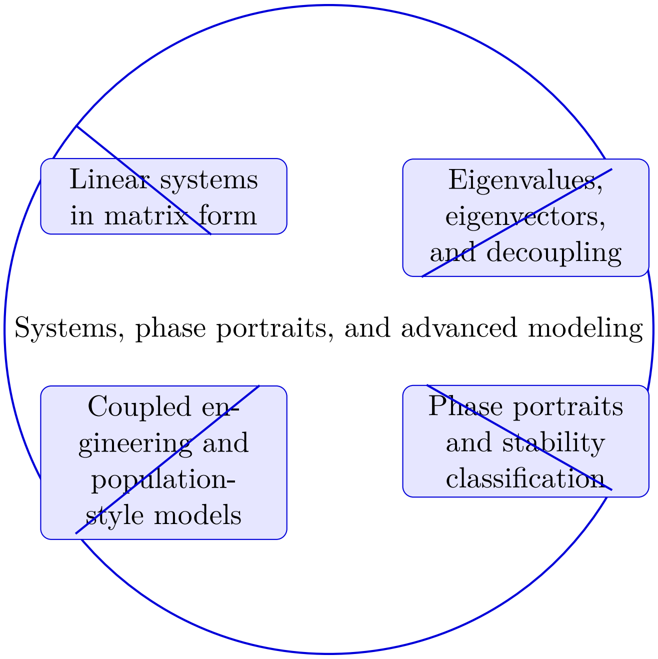
The final lesson introduces systems of equations and qualitative analysis. Students solve small linear systems, use matrix methods and eigenvalues, and inspect stability through phase portraits and equilibrium classification. Even when a full closed-form solution is available, the lesson insists on reading the geometry of trajectories and the meaning of eigenstructure.

This chapter sits at the end of Differential Equations. It develops Linear systems in matrix form, Eigenvalues, eigenvectors, and decoupling, Phase portraits and stability classification, and Coupled engineering and population-style models so that the student can move from explanation to execution without losing the thread of the course.

The central habit in this chapter is to move across words, graphs, formulas, and worked algebra without losing meaning. A correct answer is not enough on its own; the student should be able to explain why the setup is valid and how the result fits the larger mathematical structure of the course.

Core ideas

- Linear systems in matrix form
- Eigenvalues, eigenvectors, and decoupling
- Phase portraits and stability classification
- Coupled engineering and population-style models



How to think through this chapter

Problem solving in this family starts with naming the structure of the task. Students should ask which theorem, definition, or representation controls the problem before choosing a computational path. Once the structure is clear, algebraic execution should be clean, annotated, and checked against the expected behavior of the function or model.

When working this chapter, keep the following question active: @@TOKEN_0@@ A good student answer should connect setup, assumptions, and conclusion instead of only chasing a final number or sentence.

Systems of differential equations mark the point where one changing quantity is no longer enough. Real engineered systems often couple several states together, and the behavior of the whole system depends on how those states interact.

Coupling creates new behavior

A single equation can grow or oscillate, but coupled equations can transfer energy, amplify instability, or align toward an equilibrium through interaction. That is why systems feel different from earlier chapters even when each equation looks simple by itself.

Students should think of the coefficient matrix as the wiring diagram of the model. It records who influences whom and how strongly.

Eigenvalues describe the long-term personality of the system

Eigenvalues and eigenvectors can look abstract when first introduced, but in this chapter they become operational. They identify preferred directions of motion and classify whether trajectories grow, decay, rotate, or saddle away from equilibrium.

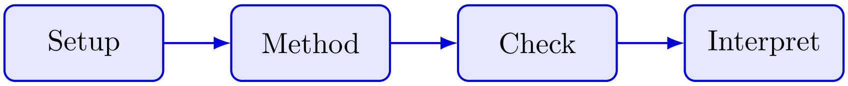
That is one of the most rewarding transitions in the course: a linear algebra object becomes a direct dynamical prediction tool.

Phase portraits teach students to read behavior without solving everything

A full symbolic solution is valuable, but not always necessary. Phase portraits let students see the geometry of motion, stability, and equilibrium structure even before every constant is solved. This is a major step toward modern dynamical-systems thinking.

It also reinforces an important engineering habit: sometimes the right question is not "what is the exact formula?" but "what behaviors are possible and which ones are stable?"

Worked example



@@TOKEN_0@@ For $x' = 2x$ and $y' = -y$, describe the origin in the phase plane.

1. The x-component grows away from zero while the y-component decays toward zero.
2. One eigenvalue is positive and one is negative.
3. That combination creates a saddle point at the origin.

Read this example twice: once for the flow of ideas and once for the technical structure of the solution.

Worked-through guided example

@@TOKEN_0@@ Find the eigenvalues of $A = [[2, 0], [0, -1]]$.

1. Notice that the matrix is diagonal.
2. Read the diagonal entries directly.
3. State the eigenvalues clearly.

Because A is diagonal, its eigenvalues are the diagonal entries 2 and -1.

Instructor commentary

Students should annotate this chapter for structure, not just facts. Mark where the argument changes direction, where the method requires a hidden assumption, and where the conclusion becomes more general than the worked example. If the chapter feels easy while you are reading it but difficult when you close the page, you have not yet converted recognition into mastery.

The most effective study pattern is read, annotate, rebuild the worked example without looking, and then solve several short-to-long problems in one sitting so the idea becomes automatic.

Practice while you read

Practice Set: Linear systems and stability

Connect matrix structure, eigen-information, and phase-plane behavior.

@@TOKEN_0@@ Find the eigenvalues of A = [[2, 0], [0, -1]].

- Hint: For a diagonal matrix, the diagonal entries are already the eigenvalues.
- Step 1: Notice that the matrix is diagonal.
- Step 2: Read the diagonal entries directly.
- Step 3: State the eigenvalues clearly.
- Checkpoint: Eigenvalues: 2 and -1

@@TOKEN_0@@ Classify the equilibrium at the origin for x' = 2x and y' = -3y.

- Hint: One positive eigenvalue and one negative eigenvalue indicate trajectories moving in opposite stability directions.
- Step 1: Write the coefficient matrix and identify its eigenvalues from the system.
- Step 2: Notice that one eigenvalue is positive and the other is negative.
- Step 3: Use the standard stability classification for that sign pattern.
- Checkpoint: The origin is a saddle point

Chapter homework

@@TOKEN_0@@ Transform techniques, step forcing, and qualitative behavior of linear systems.

1. Use Laplace transforms to solve y” + y = 0 with y(0) = 0 and y’(0) = 1.
2. Write the forcing function that turns on a constant load of 6 at t = 3 using the unit step function.
3. For the system x’ = 3x + y, y’ = x + 3y, find the eigenvalues.
4. Classify the equilibrium at the origin for x’ = -2x, y’ = -5y.

Answers for these homework problems appear in the back-of-book answer key.

Chapter summary and study notes

- Classify equilibrium behavior from eigenvalues and trace-determinant logic when appropriate.
- Use matrix notation to organize coupled dynamics cleanly.
- Explain why stable spirals, saddles, and nodes behave differently.

Study tips

- Interpret the matrix as an interaction map before doing any eigenvalue algebra.
- Classify the equilibrium from eigenvalues first, then confirm with the phase portrait.
- Use sketches to connect symbolic eigen-information to actual trajectory behavior.

Common traps

- Treating eigenvectors as algebraic leftovers instead of as meaningful directions of motion.
- Assuming every system needs a full closed-form solution before any interpretation can begin.
- Ignoring stability language after computing eigenvalues correctly.

Family-level errors to watch for

- Starting algebra before identifying the governing definition or theorem.
- Dropping notation, units, or sign conventions in the middle of a calculation.
- Treating a symbolic answer as finished without interpreting what it means.

Chapter 5

Quiz review and official exam preparation

Homework structure

- Homework Set 1: First- and second-order structure: 4 graded problems attached to chapter 1.
- Homework Set 2: Laplace methods and systems: 4 graded problems attached to chapter 2.

Quiz structure

- Quiz 1: First-order and second-order equations: 4 questions, timed, and single-attempt in the live course. Quiz 1 should be taken only after you can solve the chapter homework without outside prompts.
- Quiz 2: Laplace transforms and systems: 4 questions, timed, and single-attempt in the live course. Quiz 2 should be taken only after you can solve the chapter homework without outside prompts.

Official mastery exam

- Differential Equations cumulative mastery exam: 5 major questions, High rigor, first official attempt locks the course grade.

Differential Equations cumulative mastery exam preparation checklist

- Identify equation type before launching computation.
- Review how initial conditions interact with both time-domain and Laplace-domain methods.
- Practice resonance cases until your trial-solution adjustments are automatic.
- Be able to classify 2x2 linear systems from eigenvalues and from geometric behavior.

How to use this book before assessment

- Read the relevant chapter and rebuild both worked examples without looking.
- Solve the guided practice in the chapter before attempting the graded homework.
- Check your chapter-homework answers only after you complete a full written attempt.
- Review the quiz answer key after each chapter block and classify your errors by concept, setup, algebra, or interpretation.
- Before the official exam, revisit the chapter purposes, homework corrections, and answer-key notes rather than rereading formulas only.

Chapter 6

Course vocabulary index

- @@TOKEN_0@@: treat this as a working term in the course. You should be able to define it, recognize where it appears, and use it correctly in a solution or explanation.
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Chapter 7

Back-of-book answers and solution outlines

Guided practice answer key

Chapter 1: First-order equations and modeling

@@TOKEN_0@@

1. Solve dy/dx = 3x^2 with y(0) = 4.

- Checkpoint answer: y = x^3 + 4 - Solution note: Integrating gives y = x^3 + C. The initial condition implies C = 4, so y = x^3 + 4.

1. Solve y' + y = 0.

- Checkpoint answer: y = Ce^(-x) - Solution note: Separating gives (1/y) dy = -1 dx, so ln|y| = -x + C and y = Ce^(-x).

Chapter 2: Second-order linear equations

@@TOKEN_0@@

1. Solve y" - 5y' + 6y = 0.

- Checkpoint answer: y = C1e^(2x) + C2e^(3x) - Solution note: The characteristic equation factors as (r - 2)(r - 3) = 0, giving roots 2 and 3. Therefore y = C1e^(2x) + C2e^(3x).

1. Solve y" + 4y = 0.

- Checkpoint answer: y = C1 cos(2x) + C2 sin(2x) - Solution note: The roots are ś2i. The real-valued solution is y = C1 cos(2x) + C2 sin(2x).

Chapter 3: Laplace transforms and discontinuous forcing

@@TOKEN_0@@

1. Find L{t^2}.

- Checkpoint answer: L{t^2} = 2 / s^3 - Solution note: With n = 2, the rule gives 2! / s^3 = 2 / s^3.

1. Solve y' + y = 0 with y(0) = 3 using a Laplace transform.

- Checkpoint answer: y = 3e^(-t) - Solution note: The transformed equation gives (s + 1)Y = 3, so Y = 3/(s + 1). Inverting yields y = 3e^(-t).

Chapter 4: Systems, phase portraits, and advanced modeling

@@TOKEN_0@@

1. Find the eigenvalues of A = [[2, 0], [0, -1]].

- Checkpoint answer: Eigenvalues: 2 and -1 - Solution note: Because A is diagonal, its eigenvalues are the diagonal entries 2 and -1.

1. Classify the equilibrium at the origin for x' = 2x and y' = -3y.

- Checkpoint answer: The origin is a saddle point - Solution note: The system matrix is diag(2, -3), so the eigenvalues are 2 and -3. Opposite signs imply a saddle point, which is unstable.

Homework answer key

Homework Set 1: First- and second-order structure

1. Solve y' + 2y = e^x with y(0) = 1.

- Answer / solution summary: The integrating factor is e^(2x). Solve to get y = (1/3)e^x + (2/3)e^(-2x).

1. A tank initially holds 100 liters of pure water. Brine with concentration 0.2 kg/L flows in at 3 L/min and the well-mixed solution exits at 3 L/min. Write and solve the IVP for salt mass.

- Answer / solution summary: Let S be salt mass. Then S' = 0.6 - 0.03S with S(0)=0. Solve the linear equation for S(t).

1. Solve y" + 4y = 0 with y(0) = 2 and y'(0) = -1.

- Answer / solution summary: The solution form is y = C1 cos(2x) + C2 sin(2x). Apply the initial conditions to find constants.

1. Find a particular solution to y" - y' - 6y = 5e^(2x).

- Answer / solution summary: Since 2 is not a root of the characteristic equation, try yp = Ae^(2x) and solve for A.

Homework Set 2: Laplace methods and systems

1. Use Laplace transforms to solve y" + y = 0 with y(0) = 0 and y'(0) = 1.

- Answer / solution summary: Y(s) = 1 / (s^2 + 1), so the inverse transform gives y = sin(t).

1. Write the forcing function that turns on a constant load of 6 at t = 3 using the unit step function.

- Answer / solution summary: The forcing is 6u(t - 3).

1. For the system x' = 3x + y, y' = x + 3y, find the eigenvalues.

- Answer / solution summary: The matrix has eigenvalues 4 and 2.

1. Classify the equilibrium at the origin for x' = -2x, y' = -5y.

- Answer / solution summary: The origin is a stable node.

Quiz answer key

Quiz 1: First-order and second-order equations

1. An equation of the form y' + p(x)y = q(x) is:

- Answer key: First-order linear. That standard form defines a first-order linear equation.

1. What is the integrating factor for y' + 3y = 0?

- Answer key: Accepted answer(s): e^(3x), exp(3x), e^(3t), exp(3t). The integrating factor is e^(integral 3 dx) = e^(3x).

1. A characteristic equation with repeated real root r = 2 gives homogeneous solution:

- Answer key: C1 e^(2x) + C2 x e^(2x). Repeated roots require the second solution x e^(rx).

1. Resonance appears in undetermined coefficients when:

- Answer key: The trial solution duplicates part of the homogeneous solution. The trial must be multiplied by powers of x until linear independence is restored.

Quiz 2: Laplace transforms and systems

1. L{y"} equals:

- Answer key: s^2 Y(s) - s y(0) - y'(0). Second derivatives carry both initial position and initial slope terms.

1. If both eigenvalues of a 2x2 linear system are positive and real, the origin is usually a:

- Answer key: Unstable node. Trajectories move away from the equilibrium in both eigendirections.

1. What is the inverse Laplace transform of 1 / (s - 4)?

- Answer key: Accepted answer(s): e^(4t), exp(4t), e^(4x), exp(4x). The shift rule gives e^(4t).

1. A saddle point occurs when the linearized system has eigenvalues that:

- Answer key: Have opposite signs. Opposite signs force one stable direction and one unstable direction.

Mastery exam solution outlines

Differential Equations cumulative mastery exam

1. Solve the logistic equation P' = 0.5P(1 - P/100) with P(0) = 10 and determine the long-term population.

- What to show: Separation and partial-fraction structure; A correct carrying-capacity interpretation - Solution outline: Separate variables, integrate with partial fractions, and solve for P(t). As t approaches infinity, P approaches the carrying capacity 100.

1. Solve y" + 2y' + 5y = 0 with y(0) = 3 and y'(0) = -1.

- What to show: Characteristic roots and the correct real-valued solution form; Use of both initial conditions - Solution outline: The roots are -1 plus/minus 2i. Use y = e^(-t)(C1 cos 2t + C2 sin 2t) and apply the initial conditions.

1. Find the steady-state response for y” + 4y = 12 cos(2t) and explain why resonance changes the guess for the particular solution.

- What to show: Recognition that the forcing frequency matches the natural frequency; A corrected particular-solution form - Solution outline: Because cos(2t) already appears in the homogeneous family, use yp = t(A sin 2t + B cos 2t). Solve for the coefficients and interpret the unbounded amplitude growth in the idealized model.

1. Use Laplace transforms to solve y’ + y = u(t - 2) with y(0) = 0.

- What to show: Transform-domain equation for Y(s); A shifted inverse transform in the final answer - Solution outline: Transform to get (s + 1)Y(s) = e^(-2s) / s. Use partial fractions first, then the shift theorem to invert.

1. For the system x’ = x + 4y and y’ = 2x + y, classify the equilibrium at the origin and describe the dominant eigendirection qualitatively.

- What to show: The eigenvalues and classification; A sentence about long-term directional behavior - Solution outline: Compute eigenvalues from the matrix [[1,4],[2,1]]. One eigenvalue is positive and one is negative, so the origin is a saddle. The unstable eigendirection dominates forward-time trajectories not starting exactly on the stable line.

Reference note

For the full bibliography behind this textbook, use @@TOKEN_0@@. The answer key in this book is Summit-authored and aligned to the live course runtime.