Change, Shape, and Sum

Calculus for Classroom and Guided Self-Study

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First classroom edition

This book is a college-level calculus text covering single-variable calculus, infinite processes, differential equations, and an introductory multivariable and vector-calculus arc.

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Edition

First classroom edition.

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Table of Contents

Chapter 17. Second-Order Differential Equations and Oscillation

Part VI: Infinite Domains and Long Tails

Chapter 18. Improper Integrals and Long-Tail Behavior

Back matter

Core study tools

Practice and assessment

Extension and enrichment

Reference and lookup

Preface

Calculus is often introduced as a wall of techniques: limits, derivatives, integrals, series, partial derivatives. Students quickly learn that these topics are connected, but they do not always learn why the connections matter.

This book is built on a different promise. Every major idea in calculus can be read as an answer to one of a few durable questions:

- How is something changing?
- How much has accumulated?
- What can a small local picture tell us about a larger system?
- How can approximation become a reliable mathematical tool?

The chapters that follow aim to make those questions visible before the formal machinery takes over.

The book is deliberately concept-first but course-ready. Chapters 1-11 build the single-variable core. The later chapters extend that same local-to-global logic into multivariable calculus, vector calculus, oscillation, and improper integrals. Throughout, the goal is the same: make techniques usable without allowing them to detach from meaning.

The volume also includes a large back-of-book toolkit. That material is intentionally modular. No course is expected to use every appendix, and no reader is expected to read the support material cover to cover. Some classrooms will use only the main chapters plus homework sets and chapter tests. Others will lean on the worked examples, review banks, and oral prompts. The book is designed to allow both without changing its mathematical spine.

How to Use This Book

Each chapter is designed to work in two ways:

- as a guided self-study path for an individual reader,
- and as a class text that supports discussion, practice, and selective rigor.

Repeating elements

- `Opening question`: the mathematical situation that motivates the section.
- `Core idea`: the main concept stated in plain language.
- `Worked example`: a model of technique plus interpretation.
- `Common trap`: a mistake that is normal and worth naming early.
- `Exercises`: grouped as `Warm-up`, `Core skill`, `Interpretation`, `Challenge`, and `Modeling` so the assignment path is visible. Exercise numbers run continuously across the chapter even when the exercises are grouped by purpose.
- `Proof window`: a short justification note or selective proof. These sections are meant to explain why a result is reasonable and to name key hypotheses without turning the main text into a full proof-based course.
- `Study guide`, `worked solution atlas`, `section quizzes`, and the back-of-book practice banks: self-study support for readers who need more rehearsal than the chapter alone provides.

Three reading routes

Not every reader needs the entire support system every week. The back matter is meant to be modular.

- `Lean route`: read the chapter, work the main exercises, and use only `Answers and Hints` when you are stuck.
- `Standard route`: read the chapter, work the main exercises, and use the `Study guide`, one review appendix, and the chapter quiz or mastery bank as checkpoints.
- `Full-support route`: use the chapter plus the `Worked Solution Atlas`, review sheets, quizzes, daily skill builders, and cumulative banks when rebuilding foundations or studying independently.

If you are a strong student or working in a fast-moving course, the lean or standard route may be enough. You are not expected to read every appendix straight through.

How to read

Do not read calculus like a novel. Pause often. Before moving on:

- sketch a graph,
- explain a formula in words,
- write the units,
- and test whether you can predict what should happen before you compute it.

If a page gives you an answer but not a picture in your head, you are not done yet.

Self-study route

For independent work, use this sequence:

1. Read the `Opening question` and `Learning goals`.
2. Work through the first worked example with paper beside you.
3. Complete the `Warm-up` and `Core skill` exercises before reading further examples.
4. Use the `Worked Solution Atlas` and `Answers and Hints` only after you have written a full attempt.
5. Use the chapter quiz or mastery bank as a checkpoint before moving on.

The book is designed to support self-study, but it expects active work. Reading alone is never enough in calculus.

What to ignore until needed

The back matter is intentionally over-provisioned so different classrooms can use the same volume in different ways. That does not mean each reader needs every part immediately.

- Use the `guided examples`, `study guide`, and `answers` when you need model solutions or fast reinforcement.
- Use the `chapter tests`, `exam forms`, and `daily review` sets when assessment practice is the goal.
- Use the `historical`, `research`, and `project` appendices when you want context, enrichment, or presentation material.

If a support appendix is not helping your current goal, skip it and return later.

Rigor policy

This is an intuition-forward textbook with selective rigor, not a fully proof-based analysis text. Major formulas and theorems are motivated carefully, and key hypotheses are named when they matter to use a result correctly. The `Proof window` sections are there to connect technique to justification, to make assumptions visible, and to prepare readers for more formal work later.

For Instructors

This book is designed as a concept-first primary text for a college-level calculus sequence, with enough structure for classroom use and enough support material for guided self-study.

Course uses

- A traditional first-year calculus sequence with selected multivariable continuation.
- A concept-forward course that wants applications, proof windows, and interpretation tasks alongside computation.
- A flipped or active-learning course that uses the opening questions and exercise tiers as discussion material.
- A guided self-study or support-course setting that uses the worked solution atlas, quizzes, and mastery banks as checkpoints.

Chapter design

Each chapter includes:

- a motivating opening question,
- conceptual exposition in plain language,
- worked examples,
- common traps,
- exercise ladders from `Warm-up` through `Modeling` with chapter-wide continuous numbering,
- and a proof window that gives selective justification rather than a full proof-based treatment.

Assignment strategy

The back matter includes suggested homework sets for each chapter. A practical weekly rhythm is:

1. assign `Warm-up` and selected `Core skill` problems before the next class,
2. use `Interpretation` problems for discussion or short writing,
3. reserve `Challenge` and `Modeling` problems for quizzes, workshops, or take-home assignments.

The back matter also includes a compact study guide so students have method cues and checking habits in the same volume as the text. For courses that want more self-study support, the back matter also includes section quizzes, worked-solution models, derivation notes, and extended problem banks.

Curation principle

The back matter is deliberately modular. It is a toolkit, not a command that every course use every appendix.

Recommended default bundle for a standard course:

- main chapters,
- suggested homework sets,
- study guide,
- chapter review sheets,
- one of the quiz/test appendices,
- answers and hints.

Recommended bundle for a support-heavy or self-study course:

- default bundle,
- worked solution atlas,
- section mastery banks,
- daily review sets,
- board-style review prompts,
- cumulative review banks.

Recommended bundle for enrichment or honors use:

- default bundle,
- projects and modeling labs,
- proof and writing workshop,
- historical interludes,
- research and application excerpts,
- theory and modeling extension bank.

Classroom positioning

The book is strongest when students are repeatedly asked to move among:

- verbal interpretation,

- graphs,
- symbolic expressions,
- units,
- and contextual meaning.

Instructors who emphasize only symbolic technique will still find the text usable, but the manuscript is intentionally built to support richer mathematical sense-making.

Scope note

The early and middle chapters are written as full single-variable core instruction. The later chapters are still first-course treatments of multivariable calculus, vector calculus, oscillation, and improper integrals, but they are written to be more than a closing survey: they include worked examples, theorem-choice guidance, and explicit modeling interpretation so they can support real instruction rather than only preview it.

Notation and Conventions

Variables and functions

- Lowercase letters such as `x`, `t`, and `r` usually name input variables.
- Function names such as `f`, `g`, `A`, and `V` name relationships between quantities.
- Expressions like $f(x)$ name outputs, not multiplication.

Units

- Applied examples keep units visible whenever the interpretation matters.
- Rates are written with units per unit, such as meters per second or dollars per day.
- If units disappear during algebra, they should reappear when the answer is interpreted.

Intervals and domains

- Parentheses `(a, b)` mean endpoints are not included.
- Brackets `[a, b]` mean endpoints are included.
- Domain restrictions are stated when they matter to the mathematics.

Graphs

- Horizontal axes usually represent the input quantity.
- Vertical axes usually represent the output quantity.
- A point on a graph should be read as a statement about two linked quantities, not as decoration.

Equality and approximation

- `=` means exactly equal.
- `approx` in plain text means approximately equal.
- When numerical approximations are used, the book will usually say so explicitly.

Proof windows

Some sections contain short proof windows. They are part of the mathematics, but they do not attempt to replace a full proof-based course. Their job is to do one or more of the following:

- state the core hypotheses of a result clearly,
- justify a formula or method at an intuitive but mathematically honest level,
- connect a computational rule to an earlier theorem,
- or show the local idea behind a more formal argument.

Readers who want a first pass through the chapter may postpone these sections. Readers who want stronger rigor should treat them as mandatory waypoints.

Chapter 1. Quantities, Functions, and the Shape of Change

Opening question

A phone battery starts the morning at 18%. Twenty minutes later it is at 46%. An hour after that it is at 79%. Which interval had the faster charge: the first twenty minutes or the next hour?

This is a small question, but it already contains the heart of calculus. We are comparing change to time. We are not only asking what the battery level is; we are asking how that level behaves.

Calculus begins when we stop treating numbers as isolated facts and start treating them as parts of a changing system.

Learning goals

By the end of this chapter, you should be able to:

- describe a variable quantity and identify units,
- interpret a function in words, from a table, from a graph, and from a formula,
- compute and interpret average rate of change,
- and explain why local behavior is often simpler than global behavior.

Preview questions

Before reading further, keep these questions in view:

- What makes a changing quantity mathematically meaningful rather than just descriptive?
- Why are units not decoration but part of the calculation itself?
- When does a table or graph show enough structure to justify a model?
- Why does comparing two nearby values often teach more than staring at a single value?

1.1 Quantities that vary

In algebra, it is easy to think of a variable as just a letter. In calculus, a variable is better understood as a measurable quantity that can change.

Examples:

- time in minutes,
- distance in miles,
- temperature in degrees,
- population in thousands,
- revenue in dollars,
- height in meters.

The units matter because they tell us what a difference means. A change of `5` could mean five minutes, five liters, or five degrees. Those are not interchangeable.

When two quantities vary together, one may help predict the other. If the outside temperature changes throughout the day, the power used by a building's cooling system may change with it. If the radius of a circle changes, its area changes too. If time passes during a sprint, the runner's position changes.

Calculus studies relationships of this kind.

Units are part of the mathematics

Early in calculus, students often write the correct arithmetic and still miss the meaning because the units have disappeared. That loss matters. Suppose a water level rises from `1.2` meters to `1.8` meters in `3` minutes. The number `0.6` by itself is not the real change. The real change is `0.6 meters`, and the average rate is `0.2 meters per minute`.

That seems basic, but it scales. In economics, derivatives may be measured in `dollars per item`. In biology, rates can be `cells per hour`. In physics, an integral may accumulate `newton-meters`, `coulombs`, or `joules`. Calculus is not just symbol manipulation with letters. It is disciplined reasoning about quantities.

Example: one table, many questions

Suppose a storage tank contains the following amounts of water:

- `80` liters at $t = 0$ minutes,
- `92` liters at $t = 4$ minutes,
- `101` liters at $t = 8$ minutes.

From the same data, you can ask several different questions:

- How much water was added during the first four minutes?
- What was the average inflow rate over the entire eight-minute interval?
- Did the tank likely fill at a constant rate?

The first question asks for change. The second asks for change per unit time. The third asks whether a simple model is plausible. That layering of questions is typical in calculus.

1.2 Functions as relationships

A function is a rule or relationship that assigns exactly one output to each allowed input.

That sentence is correct, but it is too thin to be useful on its own. A more practical view is this:

> A function is a machine for describing how one quantity depends on another.

If `t` is time in hours and $T(t)$ is the temperature of coffee, then `T` tells us how temperature depends on time. If `x` is the side length of a square and $A(x)$ is its area, then `A` tells us how area depends on side length.

The input is often called the independent variable. The output is often called the dependent variable. Those names are helpful when the dependency is real, but you should not let them become magic words. The important question is always: what quantity changes, and what quantity responds?

1.3 Four ways to meet a function

You should expect to meet a function in at least four forms.

In words

"The cost of a ride is a flat boarding fee plus a charge per mile."

In a table

Miles	Cost ($)
0	3.50
2	7.10
5	12.50

In a formula

$$C(m) = 3.50 + 1.80m$$

In a graph

A graph lets your eye detect features quickly:

- increasing or decreasing behavior,
- steepness,
- turning points,
- and rough long-term trends.

No single representation is enough. Tables reveal data, formulas reveal structure, graphs reveal shape, and words reveal meaning.

One habit will matter throughout this book: whenever you see a formula, translate it back into words and units.

Domain, range, and realism

In algebra courses, domain and range are often introduced as vocabulary items. In calculus, they become modeling choices.

For the ride-cost model $C(m) = 3.50 + 1.80m$, negative mileage is not realistic even if the formula allows it algebraically. In practice, the meaningful domain is $m \geq 0$.

That point matters because later calculus techniques can produce numbers outside the realistic domain of a problem. A derivative might be negative even though a quantity itself must stay nonnegative. An optimization problem might produce a formal critical point outside the realistic interval. Good calculus keeps one eye on the formulas and one eye on the situation.

1.4 Average rate of change

Return to the battery example. The level changed from `18%` to `46%` in `20` minutes. That is a gain of `28` percentage points in `20` minutes, so the average rate of change is

$$28/20 = 1.4$$

percentage points per minute.

Over the next hour, the level changed from `46%` to `79%`. That is `33` percentage points in `60` minutes, so the average rate of change is

$$33/60 = 0.55$$

percentage points per minute.

The first interval had the faster average charge.

In general, if a quantity $f(x)$ changes as `x` moves from `a` to `b`, then the average rate of change is

$(f(b) - f(a))/(b - a)$.

This formula is one of the most important in the subject. It measures output change per unit of input change.

Example: a cooling drink

A drink cools from `88` degrees Celsius to `72` degrees Celsius in `10` minutes.

The average rate of change of temperature is

$$(72 - 88)/(10 - 0) = -16/10 = -1.6$$

degrees Celsius per minute.

The negative sign matters. It tells us the temperature is decreasing.

Why average change matters

Average rate of change is not just a warm-up for derivatives. It is the first clean measure of behavior. It already answers meaningful questions:

- How quickly is a medicine leaving the bloodstream over an hour?
- How fast is revenue increasing week to week?
- How sharply did the road climb over the last mile?

But average change also has a limit: it smooths over what happened inside the interval.

Difference quotient as a comparison machine

The expression

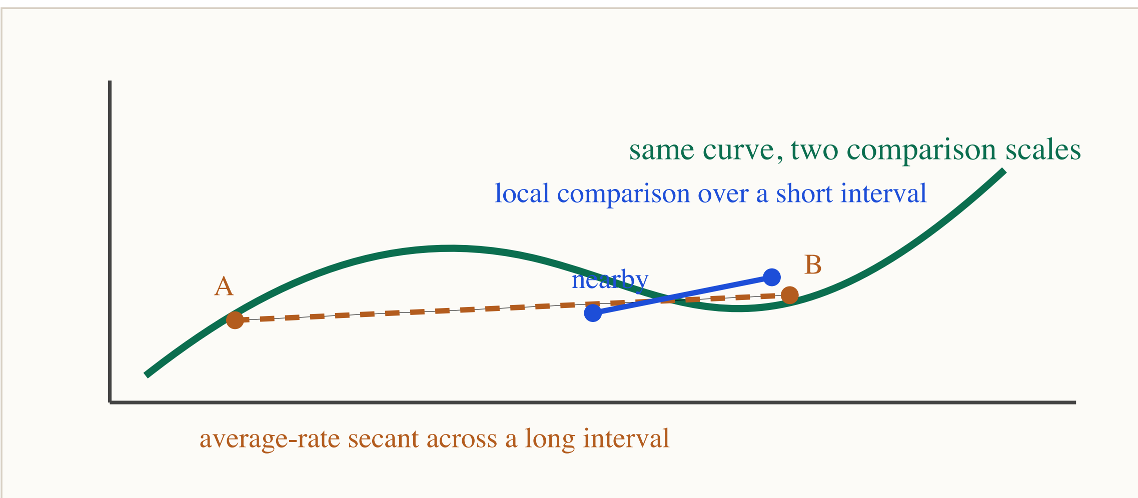
$$(f(b) - f(a))/(b - a)$$

is often called a difference quotient. The numerator records output change. The denominator records input change. The quotient packages them into a single comparison.

That structure is so central that it reappears throughout the book:

- secant slope in derivative work,
- average velocity in motion,
- average cost change in economics,
- average temperature change in heat models.

The formula itself is not deep until it is interpreted. Once interpreted, it becomes one of the main recurring templates of calculus.

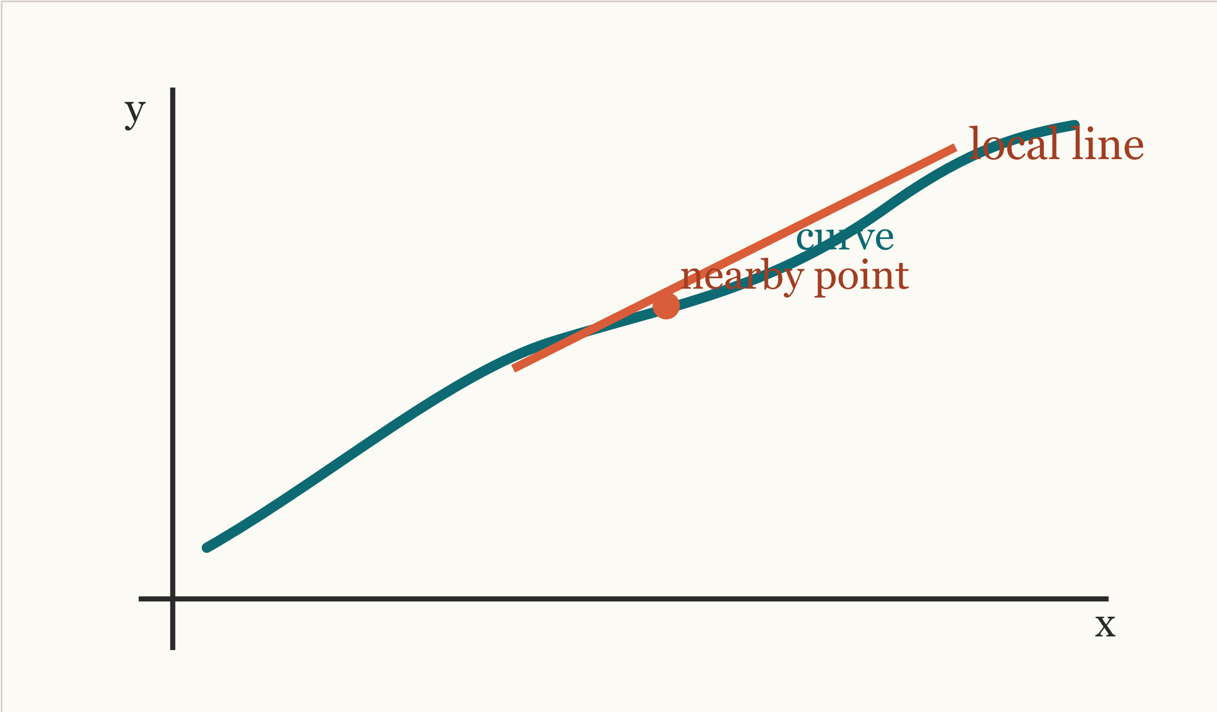


1.5 Local linear thinking

Suppose you drive on a mountain road. Over ten miles, your average speed may be `42` miles per hour. But at one sharp turn, you may have been traveling much slower. Average speed compresses an interval into one number.

Calculus wants something more local. It wants to understand behavior near a moment, not just across a long stretch.

One of the central discoveries of calculus is that many complicated functions become simpler when viewed closely. Curves often look almost straight when you zoom in enough.



That local straightness is the reason linear ideas matter so much. If a function behaves almost like a line near a point, then nearby change can be estimated using slope.

This motivates the derivative, which we will build carefully in later chapters. For now, keep the guiding idea:

> Global behavior may be curved, irregular, and complicated. Local behavior is often much simpler.

Example: area of a square

For a square with side length `s`, the area is $A(s) = s^2$.

If the side length changes from `3` to `3.1`, the area changes from `9` to `9.61`. The average rate of change over that interval is

$(9.61 - 9)/(3.1 - 3) = 0.61/0.1 = 6.1.$

If the side length changes from `3` to `3.01`, the average rate becomes

$(3.01^2 - 3^2)/0.01 = (9.0601 - 9)/0.01 = 6.01.$

Those averages are getting close to `6`. That suggests the graph of $A(s) = s^2$ behaves near $s = 3$ almost like a line of slope `6`.

We are not yet proving anything. We are training our eye to notice a pattern: shrinking the interval reveals local behavior.

Local does not mean tiny forever

The word `local` in calculus does not mean that only microscopic intervals matter. It means that to understand a function well, we often zoom in near the place of interest and ask what simpler structure appears there.

Sometimes that local structure is approximately linear. Later, near a point, it may be approximately quadratic or polynomial. But the first step is always the same: do not ask only for the whole story. Ask what the function is doing near `here`.

Historical note: tangent lines and changing quantities

Two older mathematical problems helped create calculus.

- Geometers wanted a precise way to talk about tangent lines and curvature.
- Scientists wanted a precise way to talk about instantaneous speed and continuously varying quantities.

The modern subject joins those questions together. Tangent slope and instantaneous rate are not two separate ideas. They are two languages for the same local comparison.

Quick tactics

- Before computing, name the quantities and write their units.
- Translate every formula into a sentence about dependence: what changes, and what responds?
- When a graph or table is given, look first for overall direction, steepness, and whether equal input changes produce equal output changes.
- When you compute an average rate of change, state whether it is positive, negative, or zero in words.

Common traps

- Forgetting units in a rate of change.
- Mixing up the input variable with the output quantity.
- Treating a negative rate as an error instead of meaningful information.
- Assuming a large average rate means the function was steep at every point of the interval.

Exercises

Warm-up: quantities, units, and rate meaning

1. A tank contains `140` liters of water at noon and `92` liters at `2:00 p.m.` What is the average rate of change of water volume in liters per hour?
2. The position of a cyclist is `12` kilometers from town at $t = 0$ hours and `27` kilometers from town at $t = 0.5$ hours. What is the average rate of change of position?

Core skill: average-rate computation

1. A function satisfies $f(2) = 5$ and $f(7) = 19$. Compute the average rate of change on `[2, 7]`.
2. The cost of producing `x` units is $C(x) = 800 + 12x$. Find the average rate of change of cost from $x = 20$ to $x = 100$. Interpret the answer.
3. For $A(r) = \pi r^2$, compute the average rate of change from $r = 2$ to $r = 2.5$.

Interpretation: function stories and graph language

1. Write a one-sentence interpretation of a graph that is increasing but flattening out.
2. Invent a real-world quantity pair that could reasonably be modeled by a decreasing function.

Challenge: interval behavior and counterexamples

1. A function has average rate of change `0` on an interval. Give two different possible stories for what the graph could be doing.
2. Explain why a function can have positive average rate of change on an interval even if it decreases during part of that interval.

Reflection

This chapter introduced the language of change, not yet the full machinery of calculus. If you remember only one idea, remember this one:

Calculus begins when we compare a change in output to a change in input, then ask what happens when the interval of comparison becomes very small.

Chapter 2. Nearness, Limits, and Continuity

Opening question

Imagine a formula that estimates the pressure inside a sealed container as a function of temperature. The formula works for temperatures near `20` degrees Celsius, but when you try to plug in exactly `20`, the calculator throws an error. A table of nearby values looks like this:

Temperature	**Output**
19.9	6.9
19.99	6.99
20.01	7.01
20.1	7.1

What number should you think the formula is aiming at near `20`?

This is the problem of a limit. We want language for what a function is doing near a point, even when the value at the point is hidden, missing, or misleading.

Learning goals

By the end of this chapter, you should be able to:

- explain a limit in plain language,
- estimate limits from tables and graphs,
- use basic limit laws,
- distinguish between a limit and a function value,
- and identify several common ways continuity can fail.

Preview questions

- What does it really mean for a function to get close to a number?
- Why can nearby behavior be more informative than the function's exact value at a point?
- Why do left-hand and right-hand approaches sometimes tell different stories?
- What is continuity protecting us from?

2.1 What it means to get close

The notation

$$\lim_{x \to a} f(x) = L$$

means:

> When `x` is taken close to `a`, the outputs $f(x)$ get close to `L`.

The point of this sentence is that it talks about nearby inputs, not necessarily the input `a` itself.

That distinction matters. A function can have a perfectly good limit at $x = a$ even if:

- the function is not defined at `a`,
- the function is defined at `a` but has the "wrong" value there,
- or the formula used to compute $f(x)$ becomes awkward exactly at `a`.

Limit versus value

Suppose

$f(x) = (x^2 - 1)/(x - 1)$.

If you plug in $x = 1$, you get division by zero, so $f(1)$ is not defined. But if `x` is not `1`, then

$f(x) = (x - 1)(x + 1)/(x - 1) = x + 1$.

So for values near `1`, the outputs are near `2`. That means

$$\lim_{x \to 1} f(x) = 2$$

even though $f(1)$ does not exist.

This is the first major shift in viewpoint:

> A limit is about nearby behavior, not about the exact point.

One-sided limits

Sometimes the behavior from the left and the behavior from the right are different.

- $\lim_{x \to a^-} f(x)$ means the left-hand limit.
- $\lim_{x \to a^+} f(x)$ means the right-hand limit.

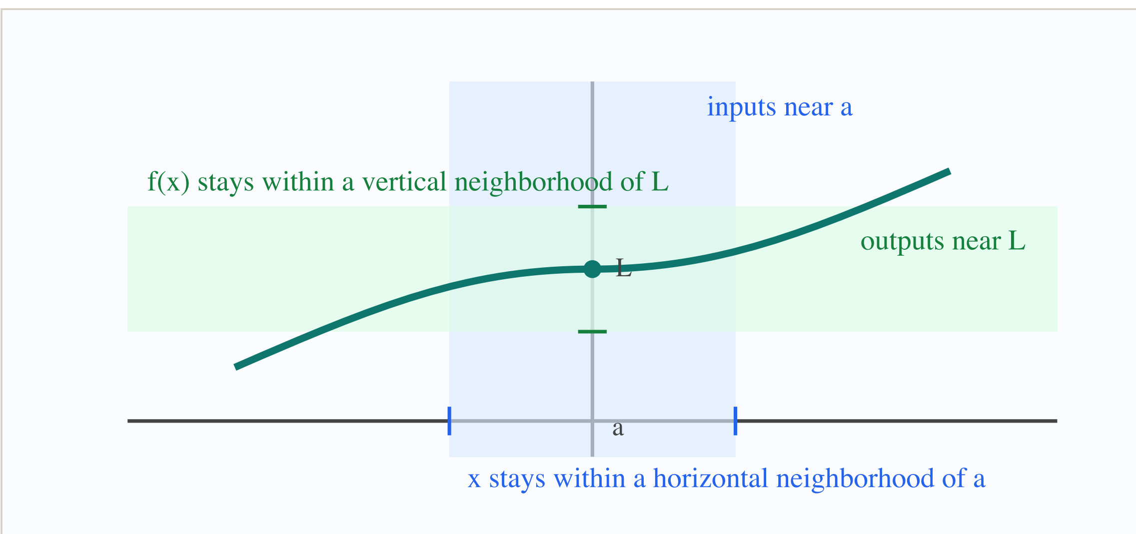
The two-sided limit $\lim_{x \to a} f(x)$ exists only when both one-sided limits exist and match.

Neighborhood language without full formalism

Later courses write limit definitions using `epsilon` and `delta`. Even before that notation, the basic picture is simple:

- keep inputs inside a horizontal band around `a`,
- watch whether outputs stay inside a vertical band around `L`.

That is the geometric skeleton of the formal definition.



2.2 Numerical and graphical evidence

Limits are often discovered before they are proved. Tables and graphs are useful because they let us see a pattern before formal rules arrive.

Example: a removable hole

Look again at

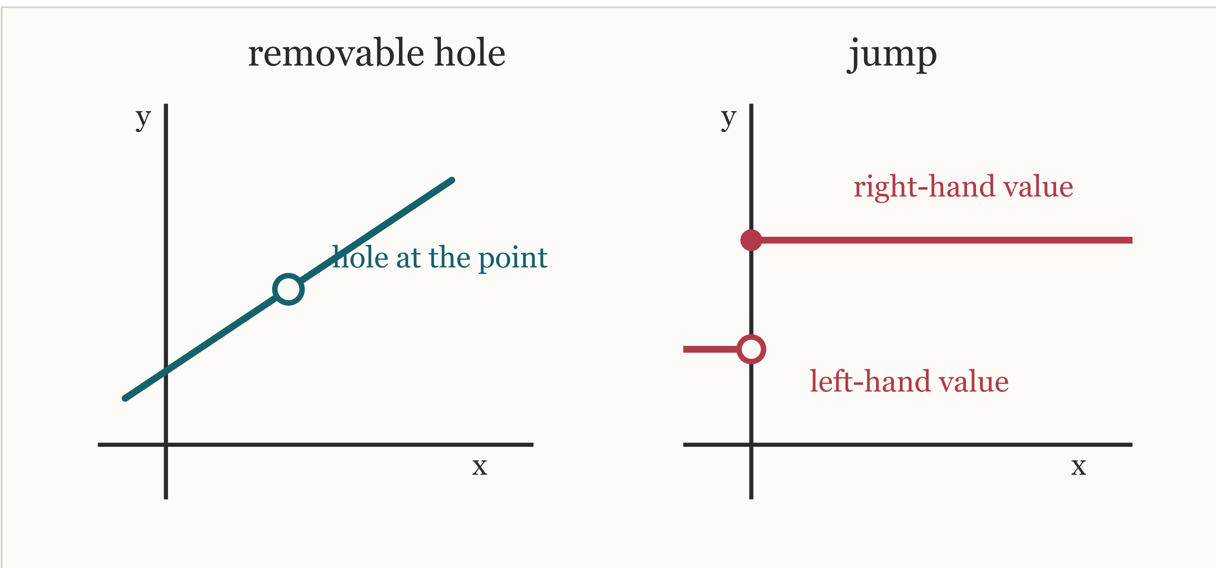
$f(x) = (x^2 - 1)/(x - 1)$.

Here is a table of values near $x = 1$:

`x`	$f(x)$
0.9	1.9
0.99	1.99
0.999	1.999
1.001	2.001
1.01	2.01
1.1	2.1

Everything in the table points toward `2`.

Graphically, the function follows the line $y = x + 1$ with one missing point at $x = 1$.



This kind of discontinuity is called a removable discontinuity. The nearby behavior is coherent; the problem is only at one point.

Example: when left and right disagree

Now define

$g(x) = -1$ for $x < 0$, and $g(x) = 2$ for $x \geq 0$.

If `x` approaches `0` from the left, the outputs stay near `-1`. If `x` approaches from the right, the outputs stay near `2`.

So:

- $\lim_{x \to 0^-} g(x) = -1$
- $\lim_{x \to 0^+} g(x) = 2$

Because those do not match, $\lim_{x \to 0} g(x)$ does not exist.

What evidence can and cannot do

Tables and graphs are strong clues, but they are not final proof. A graph can hide fine detail if the viewing window is bad. A table can miss a pattern if the chosen numbers are too coarse.

Still, numerical and graphical evidence are not second-class tools. They are often the first honest way to understand what question a calculation is asking.

Why tables need discipline

Tables are informative because they let us approach from both sides and at several scales. But a poor table can mislead. If all sampled values lie on one side of the point, the table may hide a one-sided mismatch. If the spacing is too coarse, rapid oscillation may go unnoticed.

A good calculus table usually:

- samples from the left and the right,
- uses values closer and closer to the target input,
- and avoids including the target point itself when that point is problematic.

2.3 Limit laws

Once we understand the idea of a limit, we want ways to compute limits without building a table every time.

If

- $\lim_{x \to a} f(x) = L$
- $\lim_{x \to a} g(x) = M$

then the following limit laws hold:

- $\lim_{x \to a}(f(x) + g(x)) = L + M$
- $\lim_{x \to a}(f(x) - g(x)) = L - M$
- $\lim_{x \to a}(f(x)g(x)) = LM$
- $\lim_{x \to a}(f(x)/g(x)) = L/M$, provided $M \neq 0$

The quotient rule is the only one that needs an extra hypothesis: the limiting denominator cannot be zero.

For polynomials and many rational functions, this makes limit computation almost routine.

Example: direct substitution works

Find

$\lim_{x \to 3}(x^2 + 4x - 5)$.

Because polynomials are built from sums and products, direct substitution works:

$3^2 + 4(3) - 5 = 9 + 12 - 5 = 16$.

So the limit is `16`.

Example: simplify before substituting

Find

$\lim_{x \to 1}(x^2 - 1)/(x - 1)$.

Direct substitution fails because it gives $0/0$, which is an indeterminate form. That does not mean the limit does not exist. It means the expression needs more thought.

Factor first:

$(x^2 - 1)/(x - 1) = (x - 1)(x + 1)/(x - 1) = x + 1$, for $x \neq 1$.

Now the nearby behavior is easy to read:

$\lim_{x \to 1}(x^2 - 1)/(x - 1) = 2$.

Workflow for basic limit problems

1. Try direct substitution.
2. If the result is a regular number, you are done.
3. If the result is an indeterminate form such as $0/0$, simplify the expression.
4. Check one-sided behavior if the function is piecewise or absolute-value based.

This is not the whole theory of limits, but it is a reliable start.

Toward the formal definition

Why do textbooks eventually introduce a more formal limit definition? Because pictures and tables, though powerful, are not precise enough to settle every question.

The formal definition answers this challenge:

If someone demands outputs within a chosen tolerance, can we guarantee that by taking inputs sufficiently close to the target point?

That is what the `epsilon-delta` language is trying to control. The symbols are not there to make the idea harder. They are there to make the idea exact.

2.4 Continuity as unbroken behavior

Informally, a function is continuous at a point if its graph has no tear, jump, or hole there.

More precisely, a function `f` is continuous at $x = a$ when all three conditions hold:

1. $f(a)$ exists.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

This definition says continuity is agreement between nearby behavior and actual value.

Example: a polynomial

Let $p(x) = x^3 - 2x + 4$.

Polynomials are continuous everywhere, so `p` is continuous at every real number. If `x` is near `2`, then $p(x)$ is near $p(2)$.

Example: patching a hole

Take the function

$h(x) = (x^2 - 1)/(x - 1)$ for $x \neq 1$.

This function is not continuous at $x = 1$ because $h(1)$ does not exist. But we already know the limit as `x` approaches `1` is `2`.

If we define a new function

$k(x) = (x^2 - 1)/(x - 1)$ for $x \neq 1$, and $k(1) = 2$,

then `k` becomes continuous at $x = 1$.

That is why the earlier discontinuity is called removable. The nearby behavior is already stable; one carefully chosen value repairs the break.

Continuity matters because calculus depends on it

Continuity is not only a graphing idea. It is the background condition that makes later theorems work. If a function is continuous on an interval, we can trust it not to teleport from one value to another without visiting the values in between. Later, continuity will also support tangent-line reasoning, optimization arguments, and the Fundamental Theorem of Calculus.

Continuity as reliability

In applications, continuity often expresses physical reliability. A smoothly changing sensor, a steadily expanding metal rod, or a continuously varying position function all carry the intuition that tiny input changes should not produce wild output jumps.

That intuition is not universal. Tax brackets, stepwise shipping charges, and digital threshold rules are often intentionally discontinuous. Calculus does not assume continuity everywhere. It identifies when continuity is present and exploits the consequences.

2.5 When formulas hide subtle behavior

Not all discontinuities are alike.

Holes

A hole occurs when the nearby behavior points to a value, but the function is missing that point or has the wrong value there.

Jumps

A jump occurs when the left-hand and right-hand limits exist but are different.

Examples include piecewise pricing rules, tax brackets, or simplified switching models. In real systems, those jumps may be smoothed out. In mathematical models, they can be genuine.

Infinite behavior

Sometimes a function grows without bound near a point. For example,

$$f(x) = 1/x^2$$

becomes very large as `x` approaches `0` from either side. In that case we say the function does not approach a finite real limit.

Wild oscillation

Some functions fail to settle down because they oscillate too rapidly. A famous example is $\sin(1/x)$ near $x = 0$. No matter how close you get to `0`, the outputs keep swinging through many values.

This is a useful warning:

> "Getting close in input" does not automatically force "getting close in output."

That is exactly why continuity is a condition rather than a universal law.

Historical note: why limits were invented

Limits were developed because mathematicians needed a dependable language for quantities that are approached but not necessarily reached through direct substitution. Tangent lines, instantaneous speed, infinite sums, and continuous change all demanded a notion of controlled nearness.

The modern formalism came later than the intuition. Historically, the question arrived first: what should count as the correct nearby value?

Quick tactics

- Before substituting, ask whether the problem is about two-sided, left-hand, or right-hand behavior.
- If direct substitution gives a sensible number, the limit is often straightforward. If it gives an undefined expression like $0/0$, simplify the structure before trying again.
- Keep $f(a)$ separate from $\lim_{x\to a} f(x)$ until you have evidence they agree.

Common traps

- Confusing $f(a)$ with $\lim_{x\to a} f(x)$.
- Assuming that an undefined function value means the limit does not exist.
- Believing a table proves a limit when it only suggests one.
- Ignoring one-sided behavior for piecewise functions.
- Treating $0/0$ as an answer instead of a signal to simplify.

Proof window: why factoring works in the hole example

When

$f(x) = (x^2 - 1)/(x - 1)$,

the expression is undefined at $x = 1$, but for every nearby value $x \neq 1$, it agrees exactly with `x + 1`.

That means the two formulas trace the same graph everywhere near `1` except at the point where the original formula breaks. Limits care about that nearby behavior, so replacing the expression by an equivalent one on the punctured interval is valid.

This is a general idea worth remembering:

> If two functions agree on all points sufficiently near `a` except possibly at `a` itself, then they have the same limit as $x \to a$.

Exercises

Warm-up: limit language and one-sided logic

1. In words, what does $\lim_{x\to 4} f(x) = 9$ mean?
2. For the function $g(x) = 2x + 1$, compute $\lim_{x\to 3} g(x)$.
3. If the left-hand limit at a point is `5` and the right-hand limit is `5`, what can you conclude about the two-sided limit?

Core skill: evaluating algebraic limits

1. Estimate $\lim_{x\to 2}(x^2 + 3x - 1)$.
2. Compute $\lim_{x\to -1}(x^2 - 4)/(x + 2)$.
3. Compute $\lim_{x\to 5}(3x - 2)/(x + 1)$.
4. Compute $\lim_{x\to 2}(x^2 - 4)/(x - 2)$.
5. For the piecewise function $f(x) = 1$ if $x < 3$ and $f(x) = 4$ if $x \geq 3$, determine the left-hand, right-hand, and two-sided limits at $x = 3$.

Interpretation: continuity and discontinuity

1. Describe a real-world situation where a jump discontinuity might be a reasonable first model.
2. Explain the difference between "the function has a hole" and "the function has a jump."

Challenge: constructing examples

1. Give an example of a function for which $\lim_{x \to a} f(x)$ exists but $f(a)$ is not defined.
2. Give an example of a function for which $f(a)$ exists but $\lim_{x \to a} f(x)$ does not.
3. A function is defined by $f(x) = (x^2 - 9)/(x - 3)$ for $x \neq 3$. What value should be assigned to $f(3)$ to make the function continuous?

Modeling: threshold rules and rounded outputs

1. A machine rounds shipping cost to the nearest whole dollar once the package weight reaches `10` pounds. Sketch or describe why the resulting pricing function might fail to be continuous, even if the underlying physical cost of transport changes smoothly.

Reflection

Limits teach a basic discipline of calculus: do not ask only "What is the value?" Ask also "What is the nearby behavior?" That second question will soon let us define velocity at an instant, slope at a point, and accumulation across an interval.

Chapter 3. Derivatives

Opening question

A car's odometer says the car traveled `1.2` miles between `2:00:00` and `2:01:00 p.m.` The average speed over that minute is `72` miles per hour. But was the car actually moving at `72` miles per hour at exactly `2:00:30`?

An average over a whole interval is not the same thing as behavior at an instant. The derivative is calculus' answer to that gap.

Learning goals

By the end of this chapter, you should be able to:

- explain how secant slopes lead to tangent slopes,
- compute simple derivatives from the limit definition,
- interpret the derivative as a function,
- distinguish differentiability from continuity,
- and interpret first and second derivatives in context.

Preview questions

- How does an average slope become a tangent slope?
- When should a derivative be thought of as a number, and when as a function?
- Why does the derivative carry units?
- What kinds of graph behavior destroy differentiability?

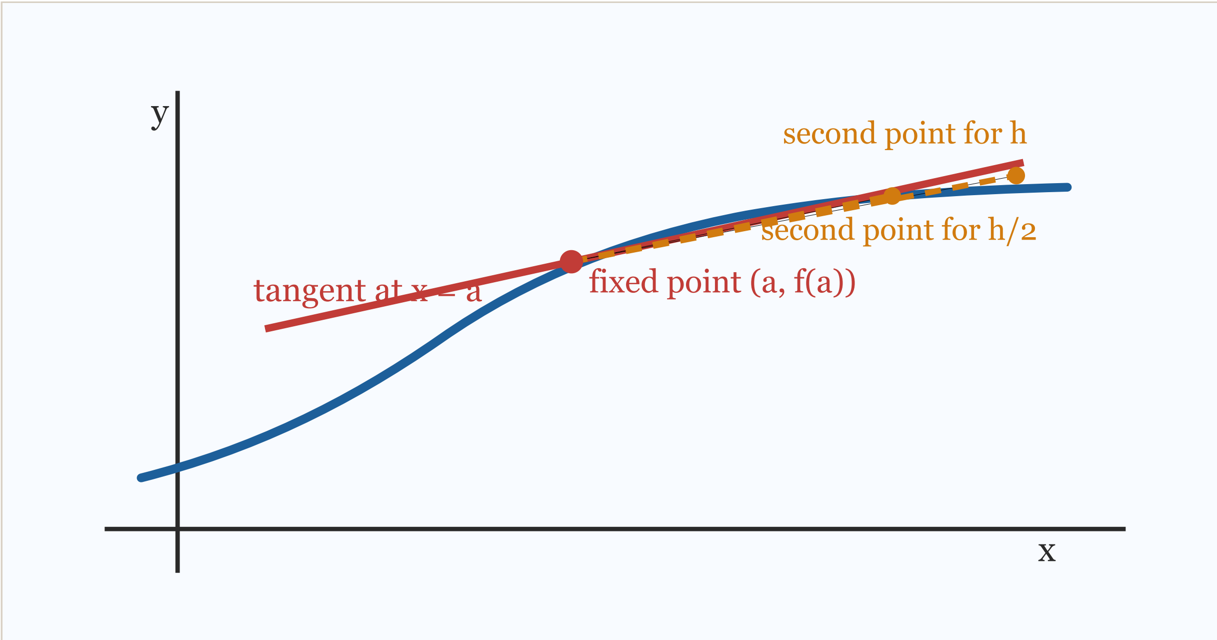
3.1 From secant lines to tangent lines

Suppose a function $s(t)$ gives position as a function of time. Over a time interval from $t = a$ to $t = a + h$, the average rate of change is

$$(s(a + h) - s(a))/h.$$

Geometrically, that is the slope of a secant line joining two points on the graph.

If we shrink the interval by making `h` smaller and smaller, the second point moves closer to the first. When the secant slopes settle toward a single number, that limiting slope is the derivative.



So the derivative of `f` at $x = a$ is

$$f\ (a) = \lim_{h \to 0}(f(a + h) - f(a))/h$$

when that limit exists.

This is a compact formula, but its meaning is geometric and physical:

- it measures the slope of the tangent line,
- it measures instantaneous rate of change,
- and it captures the best local linear description of the function.

Example: derivative of a square function at one point

Let $f(x) = x^2$. Find the derivative at $x = 3$.

Start from the definition:

$f'(3) = \lim_{h \to 0}((3 + h)^2 - 3^2)/h.$

Expand:

$((3 + h)^2 - 9)/h = (9 + 6h + h^2 - 9)/h = (6h + h^2)/h.$

For $h \neq 0$, this simplifies to

`6 + h`.

As `h` approaches `0`, the expression approaches `6`. So $f'(3) = 6$.

This matches the pattern we noticed in Chapter 1: the average rates of change near $x = 3$ were settling toward `6`.

3.2 The derivative as a new function

The derivative is not only a number attached to one point. It is often more useful to think of it as a whole new function.

If $f(x) = x^2$, then at a general input `x`, the derivative is

$f'(x) = \lim_{h\to 0}((x + h)^2 - x^2)/h.$

Expanding gives

$(x^2 + 2xh + h^2 - x^2)/h = (2xh + h^2)/h = 2x + h.$

As $h \to 0$, this approaches `2x`, so

$f'(x) = 2x.$

This single formula tells us the slope everywhere on the graph of $y = x^2$.

A derivative at a point versus the derivative function

These are different but related objects:

- `f'(3)` is one number, the slope at one input.
- `f'(x)` is a whole function that returns slopes across the domain.

Students often blur those together. The distinction matters because later problems may ask for a single instantaneous rate, the sign pattern of the derivative across an interval, or the derivative evaluated at many points.

Reading the derivative function

If `f'(x)` is:

- positive, `f` is rising locally,
- negative, `f` is falling locally,
- zero, the graph may have a horizontal tangent,
- large in magnitude, the graph is steep.

The derivative turns shape into a function you can analyze.

Example: a cubic position model

Suppose the position of a particle is

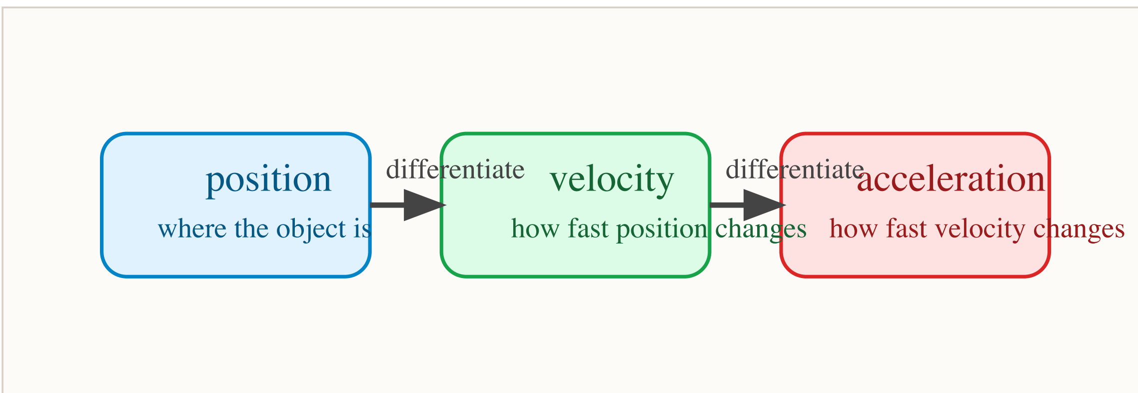
$$s(t) = t^3 - 6t^2 + 9t.$$

Then the velocity function is `s'(t)`, not just one velocity number.

We are not yet using shortcut rules, so a full derivative calculation for this cubic is postponed to the next chapter. But the important conceptual step already matters:

> The derivative is a machine that converts a position function into a velocity function.

The same idea appears in economics, biology, engineering, and geometry.



3.3 Differentiability and continuity

If a function has a derivative at a point, then it must be continuous there.

The converse is false. A function can be continuous but not differentiable.

Continuous but not differentiable

Consider $f(x) = |x|$.

The graph has no hole or jump at $x = 0$, so the function is continuous there. But the graph has a sharp corner.

- To the left of `0`, slopes are `-1`.
- To the right of `0`, slopes are `1`.

Because the one-sided tangent behavior does not match, the derivative at `0` does not exist.

This reveals an important idea:

> Continuity means no break. Differentiability means smooth enough for a stable tangent.

Other ways differentiability can fail

- corners,
- cusps,
- vertical tangents,
- and wildly oscillating graphs.

Not every nondifferentiable point looks the same, but all of them share the same problem: the local linear picture breaks down.

3.4 Rates in context

Derivatives become most meaningful when we name the quantities involved.

Position and velocity

If $s(t)$ is measured in meters and `t` in seconds, then `s'(t)` is measured in meters per second.

Revenue and marginal revenue

If $R(x)$ is revenue in dollars from selling `x` units, then `R'(x)` measures how revenue changes with output. Its units are dollars per unit.

Temperature and warming rate

If $T(t)$ is temperature in degrees Celsius after `t` minutes, then `T'(t)` tells how quickly the temperature is changing at each moment, in degrees Celsius per minute.

These interpretations all share the same grammar:

- the original function measures a quantity,
- the derivative measures how that quantity is changing with respect to the input.

Units keep the derivative honest

If a population $P(t)$ is measured in thousands of cells and `t` is measured in hours, then `P'(t)` is measured in thousands of cells per hour. If a position $s(t)$ is measured in feet and time is in seconds, then `s'(t)` is in feet per second.

That unit structure does two jobs at once:

- it tells you what the derivative means,
- and it helps catch errors.

For example, if you are differentiating a cost function and your final answer is still in plain dollars rather than dollars per unit, you should pause and check the interpretation.

Example: a runner's position

Suppose a runner's position is modeled by

$$s(t) = t^2 + 2t$$

where `s` is in meters and `t` is in seconds.

The average velocity from $t = 1$ to $t = 1 + h$ is

$((1 + h)^2 + 2(1 + h) - (1^2 + 2(1)))/h$.

Simplify:

$(1 + 2h + h^2 + 2 + 2h - 3)/h = (4h + h^2)/h = 4 + h$.

As $h \to 0$, the instantaneous velocity approaches `4` meters per second. So the derivative at $t = 1$ is `4`.

Local linear prediction

If `f'(a)` exists, then near $x = a$, the function behaves approximately like

$f(a) + f'(a)(x - a)$.

This is the beginning of linear approximation. It says the tangent line is not only a geometric object; it is a practical local prediction tool.

Derivatives answer different questions in different settings

One derivative formula can support several kinds of reasoning:

- numerical: evaluate the derivative at a point,
- graphical: use the derivative to infer rising or falling behavior,
- modeling: interpret the rate in physical or economic terms,
- predictive: build a local linear estimate near a point.

Strong calculus work moves flexibly among those interpretations instead of treating derivatives as a list of rules to memorize.

3.5 Higher derivatives

If the derivative `f'(x)` is itself differentiable, then its derivative is called the second derivative and is written `f''(x)`.

The first derivative tracks change in the original function. The second derivative tracks change in the rate of change.

Motion language

If $s(t)$ is position, then:

- `s'(t)` is velocity,
- `s''(t)` is acceleration.

Shape language

Second derivatives also connect to graph shape.

- If `f''(x)` is positive, the graph tends to bend upward.
- If `f''(x)` is negative, the graph tends to bend downward.

These ideas will become more useful when derivative rules make computation faster.

Example: a simple polynomial

If $f(x) = x^2$, then $f'(x) = 2x$, and the second derivative is

$f''(x) = 2$.

The constant positive second derivative matches the upward-opening shape of the parabola.

Second derivative as changing velocity or changing slope

The second derivative is easier to understand when attached to a story.

- In motion, the first derivative tells how position changes; the second derivative tells how velocity changes.
- In graph shape, the first derivative tracks slope; the second derivative tracks whether those slopes themselves are increasing or decreasing.

That is why second derivatives later become useful in both physics and graphing.

Historical note: derivative ideas came from several traditions

The derivative did not emerge from one single question. Geometers studied tangents, astronomers studied motion, and physicists studied changing quantities. The modern derivative unifies all of those local-change problems under one limiting process.

Quick tactics

- Estimate derivative values with short secant slopes before computing exact formulas. That keeps the meaning in view.
- Track units carefully: derivative units are always output units per input unit.
- If a graph has a corner, cusp, vertical tangent, or sudden change of rule, test differentiability before assuming a derivative exists.

Common traps

- Treating the derivative as a quotient of tiny numbers without thinking about the limiting process.
- Forgetting that the derivative depends on the input point.
- Assuming continuity automatically implies differentiability.
- Dropping units when interpreting a derivative in context.
- Thinking $f'(a) = 0$ must mean a maximum or minimum, when it may only mean a horizontal tangent.

Proof window: deriving $d/dx(x^2) = 2x$

Start from the limit definition:

$f'(x) = \lim_{h \to 0}((x + h)^2 - x^2)/h.$

Expand the square:

$(x^2 + 2xh + h^2 - x^2)/h = (2xh + h^2)/h.$

For $h \neq 0$, this becomes `2x + h`. Taking the limit as $h \to 0$ gives `2x`.

This argument matters because it models the logic of the definition. Later derivative rules are useful shortcuts, but they are trustworthy because calculations like this one lie underneath them.

Exercises

Warm-up: tangent lines and instantaneous rate meaning

1. What geometric object does the derivative represent on the graph of a function?
2. What physical quantity does the derivative of position with respect to time represent?
3. If `f'(a)` is positive, what does that usually say about the graph of `f` near $x = a$?

Core skill: limit-definition practice

1. Use the limit definition to find `f'(2)` for $f(x) = x^2$.
2. Use the limit definition to find `f'(1)` for $f(x) = x^2 + 3x$.
3. Find the average rate of change of $f(x) = x^2$ from $x = 1$ to $x = 1.1$, and compare it to `f'(1)`.
4. For $f(x) = |x|$, explain why the derivative does not exist at $x = 0$.

Interpretation: local meaning and nondifferentiability

1. Explain in words why a derivative is more local than an average rate of change.
2. Give a real-world example where the units of the derivative would be dollars per hour.

Challenge: subtle derivative logic

1. A function is continuous at $x = 4$ but not differentiable there. Sketch or describe two different ways this could happen.
2. Show from the limit definition that the derivative of a constant function is `0`.
3. A function satisfies $f'(a) = 0$. Explain why that alone does not prove $f(a)$ is a local maximum.

Modeling: motion from sampled change

1. The height of a ball is modeled by $h(t) = 20 + 18t - 4.9t^2$, with height in meters and time in seconds. Use average rates of change over short intervals near $t = 1$ to estimate the instantaneous velocity at $t = 1$. What does the sign of that velocity mean?

Reflection

The derivative takes the central idea of Chapter 1, average rate of change, and combines it with the central idea of Chapter 2, limits. The result is one of the most useful ideas in mathematics: a disciplined way to measure behavior at an instant.

Chapter 4. Working with Derivatives

Opening question

In Chapter 3, we computed derivatives from first principles. That method is honest and foundational, but it is too slow for every routine problem. If a scientist, engineer, or economist had to restart the limit definition from scratch for every new function, calculus would be nearly unusable.

This chapter develops a more practical toolkit. The goal is not to replace the meaning of the derivative, but to make that meaning easier to use.

Learning goals

By the end of this chapter, you should be able to:

- compute derivatives with the core algebraic rules,
- recognize when the chain rule is required,
- differentiate exponential, logarithmic, and trigonometric functions,
- differentiate implicitly defined relationships,
- and use tangent-line approximations for quick local estimates.

Preview questions

- How can you decide whether a new derivative problem is mainly a `product-rule` problem, a `chain-rule` problem, or both?
- Why does implicit differentiation work even when a formula has not been solved for `y`?
- When is a tangent-line estimate reliable enough to use in practice?

4.1 Power, product, quotient, and chain rules

Derivative rules are compressed versions of many limit-definition arguments. Each one saves work by capturing a pattern that appears again and again.

The power rule

For a positive integer power,

$$d/dx(x^n) = nx^(n - 1).$$

Examples:

- $d/dx(x^5) = 5x^4$
- $d/dx(x^2) = 2x$
- $d/dx(x) = 1$

This rule turns a large family of derivative calculations into one step.

Constant multiples and sums

Derivatives distribute over addition and subtraction, and constants can be pulled out:

- $d/dx(cf(x)) = cf'(x)$
- $d/dx(f(x) + g(x)) = f'(x) + g'(x)$

So if

$$f(x) = 3x^4 - 2x^2 + 7x - 9,$$

then

$$f'(x) = 12x^3 - 4x + 7.$$

The product rule

When two changing quantities are multiplied, their rates interact:

$$d/dx(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

This is not the product of derivatives. That mistake is so common that it deserves to be named directly.

Example: differentiating a product

Let

$$h(x) = x^2 \sin(x).$$

Then

$$h'(x) = 2x \sin(x) + x^2 \cos(x).$$

Each factor gets one turn being differentiated while the other is held in place.

The quotient rule

If

$q(x) = f(x)/g(x)$,

then

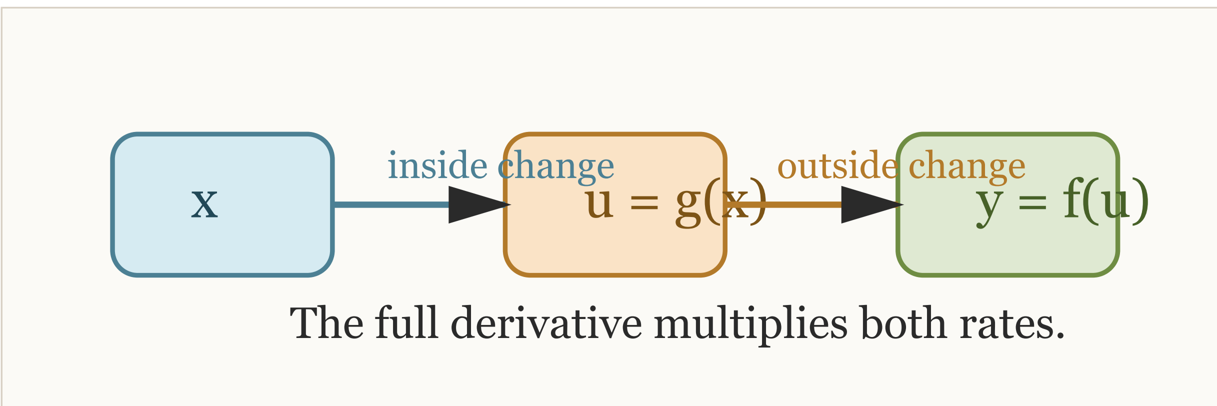
$q'(x) = (f'(x)g(x) - f(x)g'(x))/(g(x))^2$,

provided $g(x) \neq 0$.

The chain rule

The chain rule handles composition. If one quantity depends on a second quantity, and that second quantity depends on a third, then the rates multiply:

$d/dx f(g(x)) = f'(g(x))g'(x)$.



The outer function is differentiated first, but the inside is left intact until the final factor.

Example: a chain rule computation

Differentiate

$y = (3x^2 + 1)^5$.

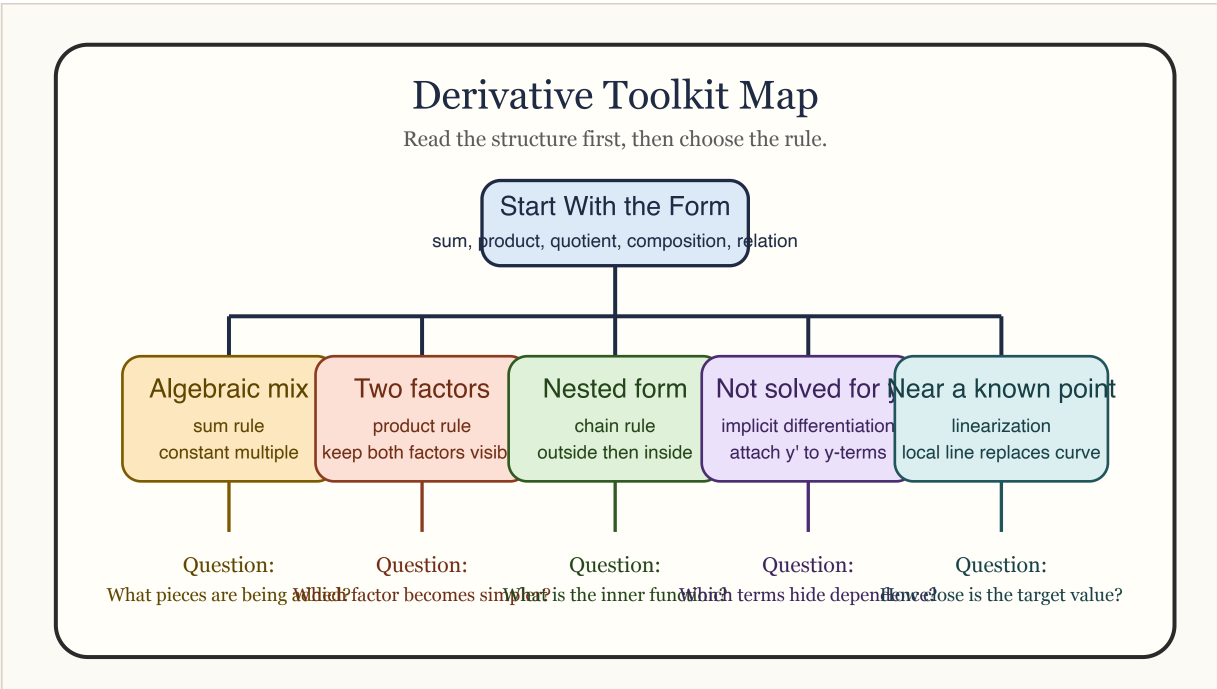
The outer function raises something to the fifth power. The inner function is $3x^2 + 1$.

So

$y' = 5(3x^2 + 1)^4(6x) = 30x(3x^2 + 1)^4$.

Workflow for complicated expressions

1. Ask whether the function is a sum, product, quotient, or composition.
2. Break it into pieces before differentiating.
3. Keep the structure visible until the derivative is assembled.
4. Simplify only after the rule has been applied correctly.



A derivative reading habit

Students often try to memorize derivative rules as separate tricks. Dense textbooks typically do something more useful: they train a reading habit. Before differentiating, pause long enough to answer three questions:

1. What is the outermost operation?
2. Which part is changing fastest?
3. Which algebraic structure should remain intact until the end?

For example, in

$$y = (x^2 + 1)^4 \sin(x^3),$$

the outermost structure is a product, but each factor contains a composition. That means the first split is the product rule, while the internal derivatives come from the chain rule. Reading structure in that order prevents a common beginner mistake: flattening the expression too early and losing the inside functions.

Worked example cluster: one idea, several surfaces

Differentiate each function and name the structural cue that matters most.

1. $f(x) = (5x - 2)^9$
2. $g(x) = x^3 e^x$
3. $h(x) = (x^2 + 1)/(x^2 - 1)$
4. $p(x) = e^{\sin x}$

The calculations are short, but the method choice is the real lesson.

For `f`, the key cue is "power of an inner expression," so

$f'(x) = 9(5x - 2)^8(5) = 45(5x - 2)^8.$

For `g`, the key cue is "product of unlike factors," so

$g'(x) = 3x^2 e^x + x^3 e^x = e^x(3x^2 + x^3).$

For `h`, the key cue is "quotient of two changing expressions," so

$h'(x) = ((2x)(x^2 - 1) - (x^2 + 1)(2x))/(x^2 - 1)^2 = -4x/(x^2 - 1)^2.$

For `p`, the key cue is "exponential with an inner angle," so

$p'(x) = e^{\sin x} \cos x.$

The point of putting these together is comparison. Strong derivative work depends less on raw speed than on correct first classification.

4.2 Derivatives of exponential and logarithmic functions

Polynomial growth is not the only kind of change we meet in the world. Populations, radioactive substances, investments, and charging curves often involve exponential or logarithmic behavior.

Exponential functions

The function e^x is special because its derivative is itself:

$d/dx(e^x) = e^x.$

More generally,

$d/dx(e^{u(x)}) = e^{u(x)} u'(x).$

For a general exponential a^x with $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}(a^x) = a^x \ln(a).$$

Logarithmic functions

For $x > 0$,

$$\frac{d}{dx}(\ln x) = 1/x.$$

With the chain rule,

$$\frac{d}{dx}(\ln(u(x))) = u'(x)/u(x),$$

provided $u(x) > 0$.

Example: an exponential model

Differentiate

$$P(t) = 500e^{0.03t}.$$

The constant `500` stays outside. The derivative of $e^{0.03t}$ is $e^{0.03t}$ times the derivative of `0.03t`, which is `0.03`.

So

$$P'(t) = 15e^{0.03t}.$$

The derivative has the same basic shape as the original function. That is one reason exponentials are so useful in growth and decay models.

Example: a logarithmic expression

Differentiate

$$y = \ln(x^2 + 4).$$

The outside function is ln, and the inside is $x^2 + 4$. By the chain rule,

$$y' = (2x)/(x^2 + 4).$$

Logarithmic differentiation as a structure-reading tool

Sometimes the main problem is not the derivative rule itself, but the awkward algebraic form of the function. Products of powers, quotients of powers, and variables in exponents can become easier after taking logarithms.

Consider

$$y = (x^2 + 1)^5 / x^3$$

for $x > 0$.

Take natural logs:

$\ln y = 5\ln(x^2 + 1) - 3\ln x.$

Differentiate both sides:

$y'/y = 5(2x)/(x^2 + 1) - 3/x.$

Then solve for `y'`:

$y' = y(10x/(x^2 + 1) - 3/x).$

Finally replace `y` with the original expression:

$y' = ((x^2 + 1)^5/x^3)(10x/(x^2 + 1) - 3/x).$

This is not the only way to differentiate the function, but it is often the cleanest. Logarithmic differentiation is especially useful later in calculus and in applied settings where formulas are built from several multiplicative ingredients.

4.3 Derivatives of trigonometric functions

Trigonometric functions carry periodic behavior: rotation, vibration, waves, and oscillation.

The basic derivative facts are:

- $d/dx(\sin x) = \cos x$
- $d/dx(\cos x) = -\sin x$

From those, together with quotient and chain rules, many others follow.

Example: a sine composition

Differentiate

$y = \sin(4x^3).$

This is a composition: sine on the outside, $4x^3$ on the inside.

So

$y' = \cos(4x^3)(12x^2) = 12x^2\cos(4x^3).$

Example: a quotient

Differentiate

$y = (x^2 + 1)/\cos x.$

Using the quotient rule,

$y' = ((2x)(\cos x) - (x^2 + 1)(-\sin x))/\cos^2 x.$

That can be left as written or simplified further if needed.

Trigonometric derivatives still tell a story

When $y = \sin x$, the derivative $\cos x$ says that the slope of the sine curve depends on where you are in the cycle.

- At a peak or trough, the slope is `0`.
- Near the midline, the slope is steep.

This is a good reminder that derivative formulas are not just symbolic output. They describe geometric behavior.

Oscillation checklist

For trigonometric functions, a derivative should preserve the basic oscillatory character of the original model. That observation gives a quick reality check:

- if the original function oscillates, the derivative should usually oscillate too,
- if the inside angle is multiplied by a constant, that constant should reappear as a scale factor,
- and if the model includes an amplitude factor, the derivative should still track that scale.

Suppose

$y = 7\cos(2x - 1).$

Then

$y' = -14\sin(2x - 1).$

The amplitude changed from `7` to `14` because the inner function changes twice as fast as `x` itself. This is the sort of interpretation engineers rely on when reading vibration or signal models.

4.4 Implicit differentiation

Not every useful relationship is solved for `y`.

The circle

$$x^2 + y^2 = 25$$

describes a curve, but it does not present `y` as a single function of `x` over the whole circle. Even so, we may want the slope of the curve.

Implicit differentiation treats `y` as a function of `x` and differentiates both sides with respect to `x`.

Example: the circle

Differentiate

$x^2 + y^2 = 25.$

Then

$2x + 2yy' = 0.$

Solving for `y'` gives

$y' = -x/y.$

This formula gives the slope at any point on the circle where $y \neq 0$.

At the point `(3, 4)`, for example,

$y' = -3/4.$

Why implicit differentiation matters

Many models relate variables without solving one variable explicitly.

Examples:

- geometric constraints,
- thermodynamic equations,
- linked rates in engineering,
- and level curves in multivariable settings.

Implicit differentiation lets the derivative machinery work even when the algebra is awkward.

Example: a product hidden inside a relation

Differentiate

$$x^2y + y^3 = 10.$$

Differentiate term by term. The first term is a product of x^2 and `y`, so

$$d/dx(x^2y) = 2xy + x^2y'.$$

The second term uses the chain rule:

$$d/dx(y^3) = 3y^2y'.$$

Therefore,

$$2xy + x^2y' + 3y^2y' = 0.$$

Group the `y'` terms:

$$(x^2 + 3y^2)y' = -2xy.$$

So

$$y' = -2xy/(x^2 + 3y^2).$$

This example is useful because it shows that implicit differentiation often combines ideas from several earlier rules at once.

Related-rates preview

Implicit differentiation becomes even more useful when several quantities depend on time. A standard classroom example is a ladder sliding down a wall. If the ladder length is fixed, then the horizontal and vertical distances are linked by the Pythagorean relation

$$x^2 + y^2 = L^2.$$

Differentiate with respect to time:

$$2x\,dx/dt + 2y\,dy/dt = 0.$$

Then

$$dy/dt = -(x/y)dx/dt.$$

The symbolic result matters less than the interpretation: a change in one dimension forces a compensating change in the other because the geometry constrains the motion.

4.5 Linear approximation

One of the most practical uses of the derivative is approximation.

If `f` is differentiable at $x = a$, then near `a`,

$f(x) \approx f(a) + f'(a)(x - a)$.

The expression on the right is the tangent-line approximation, also called the linearization of `f` at `a`.

Example: estimating a square root

Estimate $\sqrt{4.1}$ without a calculator.

Let $f(x) = \sqrt{x}$. Choose the nearby point $a = 4$, because $\sqrt{4}$ is easy.

We know:

- $f(4) = 2$
- $f'(x) = 1/(2\sqrt{x})$
- $f'(4) = 1/4$

So

$\sqrt{4.1} = f(4.1) \approx 2 + (1/4)(0.1) = 2.025$.

The true value is close to that estimate because `4.1` is close to `4`.

Example: estimating reciprocal values

Take $f(x) = 1/x$ near $x = 10$.

- $f(10) = 0.1$
- $f'(x) = -1/x^2$
- $f'(10) = -0.01$

Then

$1/10.2 \approx 0.1 + (-0.01)(0.2) = 0.098$.

This is faster than long division and reveals the local slope at the same time.

When linear approximation works well

The method works best when:

- the target input is close to the base point,
- the function is smooth there,
- and the graph is not curving too sharply over that small interval.

Linear approximation is a preview of a larger theme in calculus: difficult functions can often be replaced locally by simpler ones.

Error awareness in linearization

A linearization is not magic; it is a local agreement between a curve and its tangent line. Two checks help decide whether the estimate is trustworthy.

First, ask whether the input is genuinely close to the base point. Estimating $\sqrt{4.1}$ from $a = 4$ is sensible; estimating $\sqrt{6}$ from the same base point is much less convincing.

Second, ask whether the graph bends strongly nearby. When curvature is large, the tangent line drifts away from the graph quickly. This is why linear approximation is excellent for some engineering back-of-the-envelope calculations but poor for large extrapolations.

For

$$f(x) = 1/x$$

near $x = 10$, the graph is fairly flat, so the tangent line gives a good estimate for $1/10.2$. Near $x = 1$, the same function bends more sharply, so a comparable percentage change in input produces a noticeably worse linear estimate.

Quick tactics

- If the expression looks nested, identify the inner function before writing any derivative.
- If two large pieces are multiplied or divided, keep parentheses in place until the rule is complete.
- If `y` appears mixed with `x`, treat `y` as a dependent variable and attach `y'` whenever a `y`-term is differentiated.
- If the goal is an estimate rather than an exact value, ask whether linearization is good enough before doing heavier algebra.

Chapter review

This chapter marks the transition from foundational differentiation to fluent differentiation. The main ideas can be compressed into a short checklist:

- derivative rules encode common structural patterns,
- the chain rule is the default rule for nested change,
- exponential, logarithmic, and trigonometric functions bring their own characteristic derivative shapes,
- implicit differentiation allows slope information even when a relation is not solved explicitly,
- and tangent lines turn derivative information into fast approximations.

If a student struggles here, the usual cause is not algebra alone. It is failure to recognize structure soon enough. Reading the form of the expression is therefore as important as computing the derivative.

Mini projects

Project 1: build a derivative decision chart

Choose twenty functions from this chapter and classify each as primarily requiring the sum rule, product rule, quotient rule, chain rule, implicit differentiation, or linearization. Then identify which functions require more than one idea. The goal is to build a personal decision chart that reflects actual examples rather than memorized slogans.

Project 2: tangent-line estimation in the wild

Find five quantities that people often estimate mentally, such as square roots, reciprocals, or exponential growth factors. For each one, choose a nearby easy base point, produce a linear approximation, compare it with the exact value, and describe why the approximation is or is not effective.

Common traps

- Treating the derivative of a product as the product of derivatives.
- Forgetting the inside derivative in a chain rule problem.
- Using logarithmic derivative formulas without checking the domain.
- Differentiating `y` implicitly and forgetting the factor of `y'`.
- Using linear approximation too far from the base point.

Proof window: why the chain rule is plausible

Suppose $y = f(u)$ and $u = g(x)$. If a small change in `x` produces a small change in `u`, and that small change in `u` produces a small change in `y`, then the overall rate of change from `x` to `y` should combine those two effects.

In symbols, that suggests

$dy/dx = (dy/du)(du/dx)$.

This is not a full proof, but it captures the right structure. A rigorous proof uses the limit definition and careful error control. The important conceptual point is that composition creates a rate-inside-a-rate situation, so multiplication of rates is natural.

Exercises

Warm-up: basic derivative rules

1. Differentiate x^6.
2. Differentiate $3x^4 - 7x + 2$.
3. What extra factor appears when you differentiate $\sin(5x)$?

Core skill: products, quotients, and compositions

1. Differentiate $f(x) = (x^2 + 1)(x^3 - 2)$.
2. Differentiate $g(x) = (x^2 + 4)/(x - 1)$.
3. Differentiate $h(x) = (2x - 3)^7$.
4. Differentiate $p(x) = e^{4x}$.
5. Differentiate $q(x) = \ln(x^2 + 9)$.
6. Differentiate $r(x) = \cos(3x^2)$.

Interpretation: structure and local approximation

1. In words, explain why the chain rule is needed for $\sqrt{1 + x^4}$.
2. Explain why linear approximation is a local idea rather than a global one.

Challenge: implicit and mixed structures

1. Differentiate $y = x^3e^x$.
2. The curve $x^2 + xy + y^2 = 7$ is given implicitly. Find dy/dx.
3. Find the linearization of $f(x) = x^3$ at $x = 2$ and use it to estimate 2.05^3.

Modeling: chain-rule rates in context

1. The radius of a circular oil spill changes with time according to $r(t) = 5 + 0.2t$, where `r` is in meters and `t` in hours. The area is $A = \pi r^2$. Use the chain rule to find how quickly the area is changing at $t = 3$.

Reflection

This chapter turns the derivative from a definition into a working language. The real skill is not only calculation. It is seeing the structure of a function well enough to choose the right derivative tool and interpret the result.

Chapter 5. What Derivatives Tell Us

Opening question

A drone climbs from a rooftop, levels off for a moment, then begins to descend. Suppose you have no video of its flight path. You are given only a graph of its vertical velocity.

Could you still tell when the drone was rising, when it was falling, and when it reached a highest point?

This chapter is about reading behavior from derivatives. Once a derivative has been found, it becomes evidence about the original function.

Learning goals

By the end of this chapter, you should be able to:

- use the sign of the derivative to analyze increasing and decreasing behavior,
- identify critical points and classify local extrema,
- interpret second derivatives in terms of concavity,
- build a graph from derivative evidence,
- and set up basic optimization problems from context.

Preview questions

- Why does a derivative sign chart reveal interval behavior more reliably than checking a few function values?
- How can a critical point fail to be either a maximum or a minimum?
- Why do many optimization problems reduce to one variable only after a constraint is used?

5.1 Increasing and decreasing behavior

If `f'(x)` is positive at a point, the graph of `f` is tilting upward there. If `f'(x)` is negative, the graph is tilting downward.

This suggests a simple reading rule:

- positive derivative means local increase,
- negative derivative means local decrease.

Example: a cubic function

Take

$$f(x) = x^3 - 3x.$$

Its derivative is

$$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1).$$

Now analyze the sign of `f'(x)`:

- if $x < -1$, both factors are negative, so `f'(x)` is positive;
- if $-1 < x < 1$, one factor is negative and one is positive, so `f'(x)` is negative;
- if $x > 1$, both factors are positive, so `f'(x)` is positive.

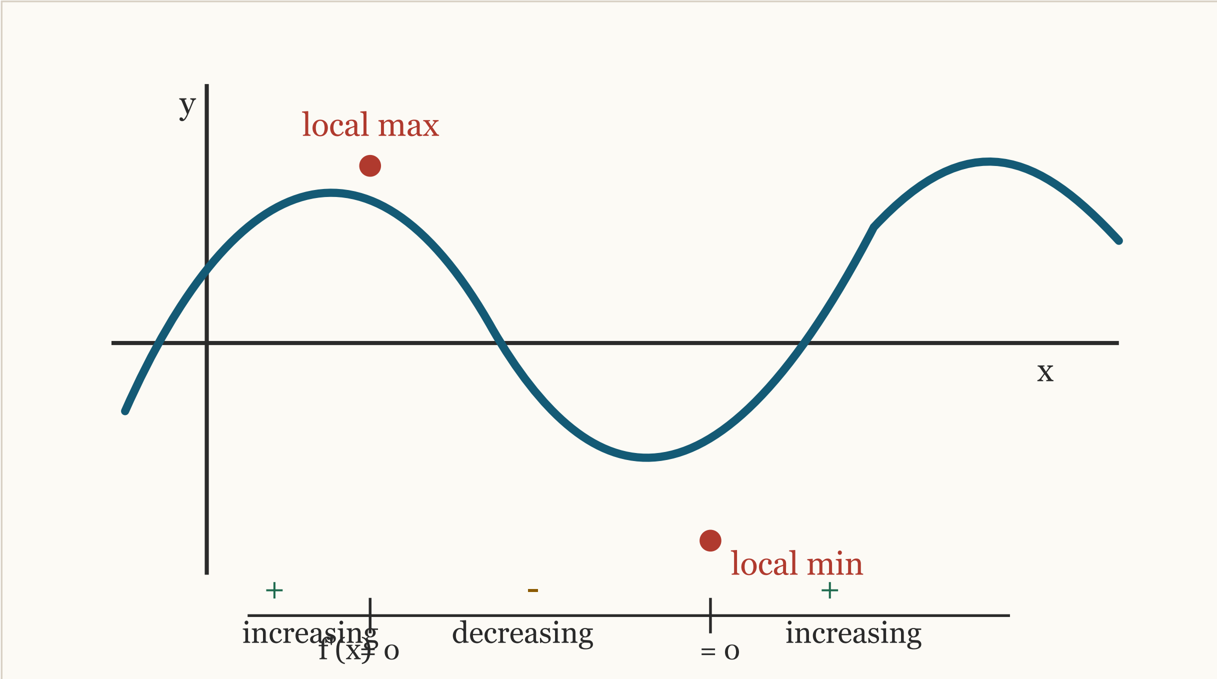
So the function:

- increases on `(-infinity, -1)`,
- decreases on `(-1, 1)`,
- increases on `(1, infinity)`.

This tells us much more than the formula alone. It tells the story of the graph.

Sign charts

A sign chart is a compact way to organize this information.



The sign chart is not decoration. It is a thinking tool. It keeps the derivative separate from the original function while showing how one controls the other.

A warning about isolated points

Knowing that $f'(a) = 0$ at a single point does not tell us enough. The derivative must be studied on both sides of the point. Behavior is an interval story, not just a point story.

Reading signs from factors

The factorized form

$$f\ (x) = 3(x - 1)(x + 1)$$

is valuable because each factor changes sign only at one critical location. In practice, this is why derivative analysis so often begins with factoring or isolating a denominator sign. The sign chart becomes far easier to build when the derivative is written as a product of simple pieces.

5.2 Critical points and local extrema

A critical point of `f` is a point in the domain where either:

- $f'(x) = 0$, or
- `f'(x)` does not exist.

These points matter because local maxima and minima often occur there.

Example: local maximum and minimum

Return to

$f(x) = x^3 - 3x.$

Its critical points are $x = -1$ and $x = 1$, because those are the points where $f'(x) = 0$.

Now use the sign changes:

- At $x = -1$, the derivative changes from positive to negative, so `f` has a local maximum there.
- At $x = 1$, the derivative changes from negative to positive, so `f` has a local minimum there.

Compute the values:

- $f(-1) = (-1)^3 - 3(-1) = 2$
- $f(1) = 1 - 3 = -2$

So the graph has a local maximum at `(-1, 2)` and a local minimum at `(1, -2)`.

First derivative test

The first derivative test is the sign-change rule:

- `+` to `-` means local maximum,
- `-` to `+` means local minimum,
- no sign change means no local extremum.

Example: a flat point that is not an extremum

Let

$f(x) = x^3.$

Then

$f'(x) = 3x^2.$

At $x = 0$, the derivative is `0`, so `0` is a critical point. But $3x^2$ is never negative. The derivative is positive on both sides except at the point itself, so the function keeps increasing through `0`.

Therefore $x = 0$ is not a maximum or minimum. It is a flat point, not a turning point.

This is why derivative sign changes matter more than the equation $f'(x) = 0$ by itself.

Second derivative test

When the first derivative identifies a critical point and the second derivative is easy to compute, the second derivative test can classify the point quickly:

- if $f'(c) = 0$ and $f''(c) > 0$, the graph is locally concave up, so $f(c)$ is a local minimum;
- if $f'(c) = 0$ and $f''(c) < 0$, the graph is locally concave down, so $f(c)$ is a local maximum;
- if $f''(c) = 0$, the test is inconclusive.

This test is efficient, but it should not replace the first derivative test in every situation. The sign change of `f'` is often more informative because it directly records interval behavior.

Example: using the second derivative test

Let

$$f(x) = x^4 - 4x^2.$$

Then

$$f'(x) = 4x^3 - 8x = 4x(x^2 - 2),$$

so the critical points are $x = 0$, $x = \sqrt{2}$, and $x = -\sqrt{2}$.

Now compute

$$f''(x) = 12x^2 - 8.$$

At $x = 0$, we have $f''(0) = -8$, so $x = 0$ is a local maximum.

At $x = \pm\sqrt{2}$, we have $f''(\pm\sqrt{2}) = 16$, so both are local minima.

The second derivative test turns a long sign analysis into a short classification when the algebra behaves nicely.

5.3 Concavity and inflection

The first derivative describes whether a function is rising or falling. The second derivative describes how that rise or fall is itself changing.

If the slopes are increasing, the graph bends upward. If the slopes are decreasing, the graph bends downward.

- $f''(x) > 0$ suggests concave up.
- $f''(x) < 0$ suggests concave down.

Example: a quadratic

For

$f(x) = x^2$,

we have

- $f'(x) = 2x$
- $f''(x) = 2$

Because the second derivative is always positive, the graph is concave up everywhere.

Example: a cubic with changing concavity

Let

$f(x) = x^3$.

Then

- $f'(x) = 3x^2$
- $f''(x) = 6x$

So:

- for $x < 0$, $f''(x) < 0$, so the graph is concave down;
- for $x > 0$, $f''(x) > 0$, so the graph is concave up.

The change occurs at $x = 0$, which is an inflection point.

What concavity means visually

Concavity is about how tangent slopes evolve:

- on a concave-up graph, tangent slopes tend to rise as you move left to right;
- on a concave-down graph, tangent slopes tend to fall.

That viewpoint often helps more than memorizing vocabulary alone.

Inflection points require a change

It is common to solve $f''(x) = 0$ and announce an inflection point too early. An inflection point requires a real change in concavity, not merely a second derivative of zero. The logic is parallel to the first derivative story:

- $f''(c) = 0$ identifies a candidate,

- a sign change in `f''` confirms the change in bending.

5.4 Graphing from derivative evidence

When you analyze a function, you are usually combining several layers of information:

1. domain,
2. intercepts,
3. critical points,
4. increasing/decreasing intervals,
5. concavity,
6. end behavior.

This chapter focuses on the derivative layers, but good graphing is always a synthesis.

Graphing workflow

1. Compute `f'(x)` and locate critical points.
2. Build a sign chart for `f'(x)`.
3. Use the sign chart to locate increasing/decreasing intervals and local extrema.
4. Compute `f''(x)` when useful.
5. Use `f''` to study concavity and possible inflection points.
6. Place all the evidence on one sketch.

Example: sketching a rational function

Consider

$f(x) = x/(x^2 + 1)$.

Differentiate:

$f'(x) = ((1)(x^2 + 1) - x(2x))/(x^2 + 1)^2 = (1 - x^2)/(x^2 + 1)^2$.

The denominator is always positive, so the sign is controlled by $1 - x^2$.

Critical points occur at $x = -1$ and $x = 1$.

- $f'(x) < 0$ for $x < -1$
- $f'(x) > 0$ for $-1 < x < 1$
- $f'(x) < 0$ for $x > 1$

So the graph decreases, then increases, then decreases. That means:

- local minimum at $x = -1$,

- local maximum at $x = 1$.

Compute the values:

- $f(-1) = -1/2$
- $f(1) = 1/2$

Already the graph is mostly determined.

Second derivative as refinement

The second derivative is often best used to refine a graph, not to replace the first derivative analysis. It helps you decide how the curve bends as it passes through the main features.

Asymptotes and derivative evidence

For rational functions, good graphing also uses asymptotic information. A vertical asymptote may break the graph into separate intervals, and the derivative sign must then be interpreted on each interval separately. A horizontal or oblique asymptote gives end-behavior guidance that the local derivative alone does not provide.

This is a reminder that graphing from derivatives is a synthesis problem. The derivative does not erase everything learned earlier about domain, continuity, or asymptotic behavior.

5.5 Optimization and modeling

Optimization asks for the best possible value of a quantity under given conditions.

Words like

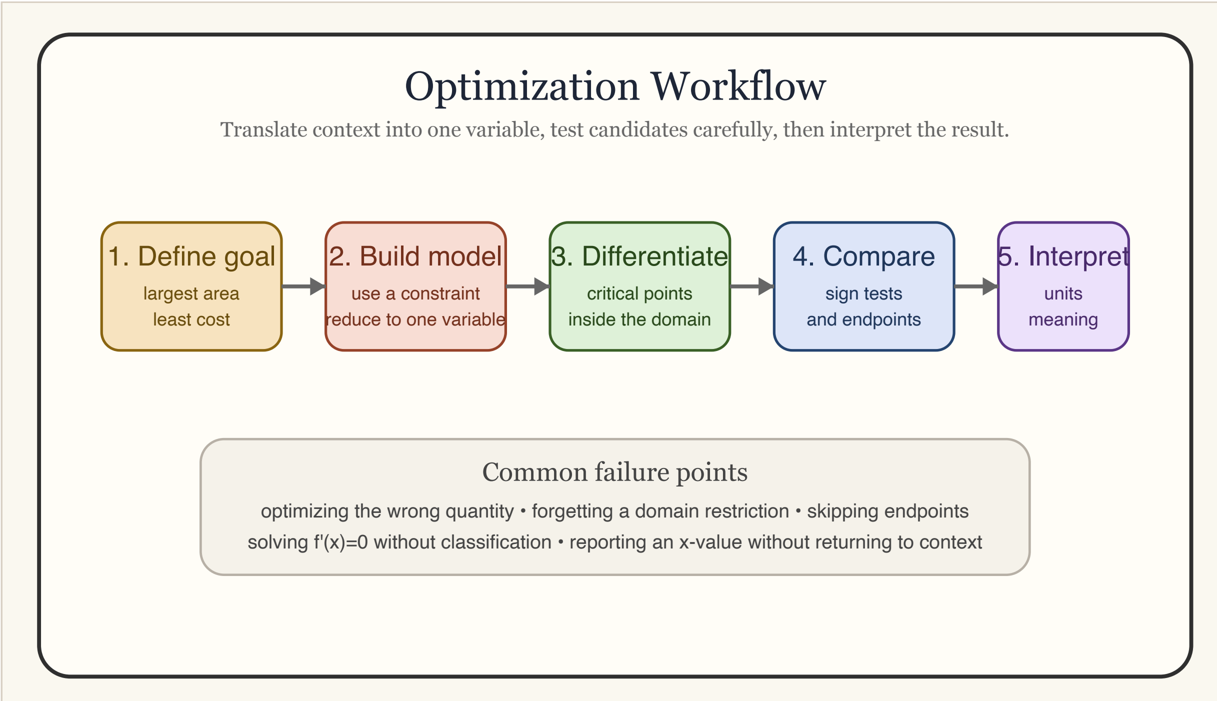
- maximize,
- minimize,
- largest,
- smallest,
- least cost,
- greatest area

are optimization signals.

General optimization workflow

1. Identify the quantity to optimize.
2. Express it as a function of one variable.
3. Determine the realistic domain.

4. Differentiate and find critical points.
5. Compare candidates using derivative tests or endpoint values.
6. Interpret the result in the original context and units.



Example: the largest rectangle with fixed perimeter

Suppose a rectangle has perimeter `40` meters. Which dimensions give the greatest area?

Let the side lengths be `x` and `y`. Then:

$$2x + 2y = 40,$$

so

$$y = 20 - x.$$

The area is

$$A(x) = x(20 - x) = 20x - x^2.$$

Differentiate:

$$A'(x) = 20 - 2x.$$

Set the derivative equal to `0`:

$$20 - 2x = 0, \text{ so } x = 10.$$

Then $y = 10$, so the maximizing rectangle is a square.

Because

$$A''(x) = -2,$$

the graph is concave down everywhere, confirming the critical point gives a maximum.

Example: minimizing material

Optimization is not always geometric. A manufacturer might minimize metal, fuel, or cost. A biologist might minimize error in a fitted parameter. A city might optimize timing between traffic lights. The calculus structure is the same: build a quantity function, differentiate, and interpret.

Closed-interval optimization

In many realistic problems, the domain is bounded. That means the best value may occur at:

- an interior critical point,
- the left endpoint,
- or the right endpoint.

This is one reason optimization work cannot stop at solving $f'(x) = 0$. If the problem is posed on a closed interval, endpoints are genuine candidates and must be checked.

Example: a fenced pasture along a river

Suppose `200` meters of fencing are available to enclose a rectangular pasture that uses a straight river as one side, so only three sides require fencing.

Let `x` be the width perpendicular to the river and `y` the length along the river. Then

$$2x + y = 200,$$

so

$$y = 200 - 2x.$$

The area is

$$A(x) = x(200 - 2x) = 200x - 2x^2.$$

Differentiate:

$$A'(x) = 200 - 4x.$$

Set the derivative equal to zero:

$200 - 4x = 0,$

so $x = 50$. Then $y = 100$.

Because $A''(x) = -4$, the area function is concave down, confirming a maximum. The best pasture has width `50` meters and length `100` meters.

Why optimization feels difficult

Most optimization difficulty occurs before differentiation:

- deciding what the variable should be,
- identifying the actual quantity to optimize,
- and expressing the constraint correctly.

Once the model is clean, the calculus is often routine. That is why textbooks devote so much space to setup and interpretation rather than only to derivative mechanics.

Quick tactics

- Build sign charts from intervals, not from isolated points.
- Treat $f'(x) = 0$ and $f''(x) = 0$ as candidate generators, not final conclusions.
- In graphing, combine derivative evidence with domain, continuity, and asymptotes.
- In optimization, write the constraint early and reduce to one variable before differentiating.

Chapter review

This chapter teaches one of the subject's most important habits: a derivative is evidence. It tells us:

- where a function rises or falls,
- where local extremes are plausible,
- how the graph bends,
- and which candidate solutions in an optimization problem deserve attention.

The power of derivatives becomes much clearer here than in a formula-only chapter. Once a derivative has been computed, it becomes a reading instrument for the original function.

Mini projects

Project 1: graph from evidence only

Choose a function and hide its original graph. Provide only domain information, a sign chart for `f'`, a sign chart for `f''`, and a few anchor values. Reconstruct the most plausible graph and then compare it with the actual graph.

Project 2: optimize a design brief

Write a one-page design brief for an optimization problem from packaging, agriculture, traffic planning, or manufacturing. Include the constraint, the quantity to optimize, the resulting one-variable model, and a short discussion of why the winning answer makes sense in context.

Common traps

- Solving $f'(x) = 0$ and assuming every solution is a max or min.
- Forgetting critical points where the derivative does not exist.
- Using the second derivative without understanding what quantity is being optimized.
- Ignoring endpoints in a closed-interval problem.
- Reporting an answer without units or context.

Proof window: why sign changes matter

If `f'(x)` is positive on an interval, then small steps to the right tend to produce positive output change, so the function rises across that interval. If `f'(x)` is negative, small rightward steps tend to produce negative output change, so the function falls.

This chapter uses that principle as a working rule. A fully rigorous justification is often proved with the Mean Value Theorem, but the sign chart already captures the correct local logic: derivative sign records the direction of change.

Exercises

Warm-up: sign and critical-point meaning

1. If $f'(x) > 0$ on an interval, what does that say about `f` there?
2. What is a critical point?
3. If $f''(x) > 0$, what does that usually suggest about the graph?

Core skill: first- and second-derivative analysis

1. For $f(x) = x^3 - 3x$, find the intervals where `f` increases and decreases.
2. For $g(x) = x^4 - 4x^2$, find all critical points.
3. For $h(x) = x^3$, determine whether $x = 0$ is a local extremum.
4. For $p(x) = x^2 + 4x + 1$, determine the concavity.
5. For $q(x) = x/(x^2 + 1)$, find the local extrema.

Interpretation: graph behavior from derivative evidence

1. Explain in words how the second derivative is related to changing slope.
2. Describe the difference between a critical point and an inflection point.

Challenge: subtle extrema and inflection logic

1. Give an example of a function whose derivative is zero at a point but which has no extremum there.
2. Give an example of a function with a local minimum where the derivative does not exist.
3. A function is increasing everywhere but changes concavity at one point. Describe a possible graph.

Modeling: optimization

1. A farmer has 120 meters of fencing to enclose a rectangular pen against a straight river, so only three sides need fencing. What dimensions maximize the area?
2. The profit from selling `x` units is modeled by $P(x) = -0.2x^2 + 24x - 100$. Find the production level that maximizes profit.

Reflection

Derivatives are not only numbers to compute. They are descriptions of behavior. A derivative tells you how the graph is moving now, and from that local evidence you can reconstruct much of the global picture.

Chapter 6. The Integral as Accumulation

Opening question

A tank is filling with water, but the inflow is not constant. At first the water enters slowly, then faster, then slowly again. A sensor records the inflow rate in liters per minute.

How can we recover the total amount of water added over the next hour?

This is the central question of integration. When a rate changes, totals are built by accumulation.

Learning goals

By the end of this chapter, you should be able to:

- interpret a definite integral as accumulated change,
- distinguish signed area from ordinary geometric area,
- build Riemann sums from small pieces,
- explain the role of the definite integral in net change,
- and interpret the average value of a function on an interval.

Preview questions

- If a rate is changing, what should the small pieces of total accumulation look like?
- Why is the definite integral about more than geometric area?
- How do signed contributions differ from total amount?
- Why does average value depend on the integral rather than just endpoint arithmetic?

Foundations note: area comes before area pictures

This chapter uses area language because ordinary geometric area already exists for simple regions such as squares, rectangles, and triangles. In that setting, area is itself a function on a class of planar regions, with normalization, congruence invariance, and finite additivity as its organizing properties.

The definite integral does not create geometric area from nothing. It extends the same additive logic to continuously varying slices and to signed-accumulation questions such as displacement and net change.

Appendix S makes this viewpoint explicit and also notes that not every bounded set has Jordan area.

6.1 Net change and signed area

Suppose $r(t)$ is the rate at which water enters a tank, measured in liters per minute. If the rate stayed constant at `5` liters per minute for `10` minutes, the total added water would be

$$(5 \text{liters}/\text{minute})(10 \text{minutes}) = 50 \text{liters}.$$

That is simple because the rate is constant.

But when the rate varies, the same unit logic still guides us. A small slice of accumulated amount is approximately

`(rate)(small time width)`.

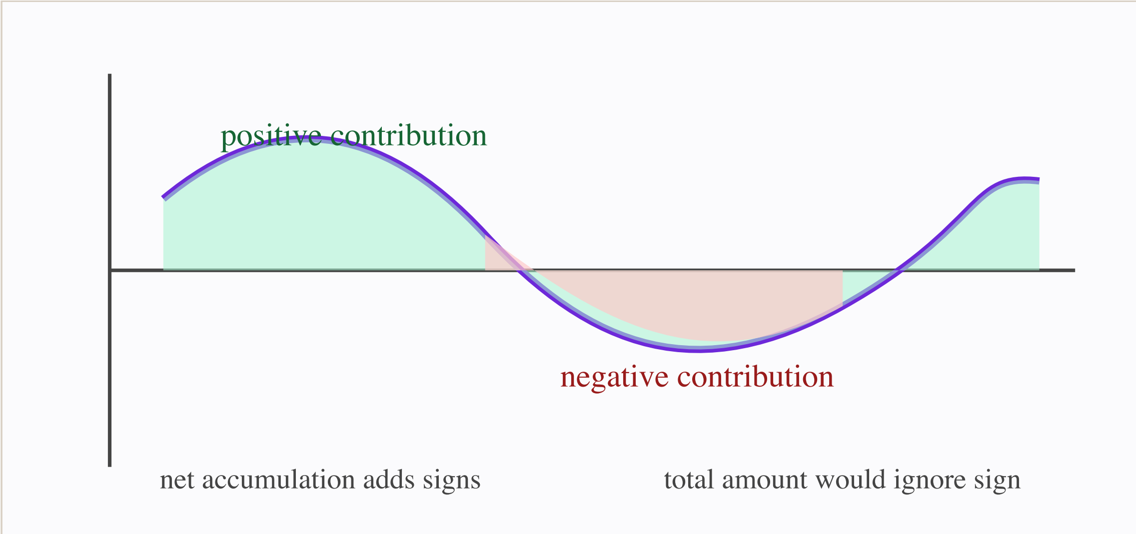
Add many such slices and you recover the total.

Signed area

When a function is graphed against its input, accumulation often appears visually as area-like regions. But the crucial word is not area. It is signed.

- Above the axis, contributions are positive.
- Below the axis, contributions are negative.

This is why definite integrals naturally express net change.



Example: positive and negative velocity

If $v(t)$ is velocity, then positive values correspond to motion in one direction and negative values to motion in the opposite direction.

The definite integral of velocity over a time interval gives displacement, not total distance traveled. Negative motion subtracts from positive motion.

Example: a simple net-change calculation

Suppose a machine fills a container at `3` liters per minute for `4` minutes, then drains it at `1` liter per minute for `2` minutes.

Net change:

$(3)(4) + (-1)(2) = 12 - 2 = 10 \text{liters}.$

That is the basic logic of signed accumulation. Later, the definite integral will let us do the same thing when the rate varies continuously.

6.2 Riemann sums

The exact integral is built from approximate sums.

Accumulation is a product story

At the level of units, integration repeatedly uses the same idea:

- $(\text{velocity})(\text{time}) = \text{displacement},$
- $(\text{density})(\text{length}) = \text{mass},$
- $(\text{power})(\text{time}) = \text{energy},$

- $(\text{priceperpound})(\text{pounds}) = \text{cost}.$

That product structure matters more than the notation at first. When students lose the product story, integrals start looking arbitrary. When they keep it, many application problems become variations of one template.

Take an interval `[a, b]` and split it into many short pieces of width Δx. On each piece, choose a sample point c_i. Then the small accumulated contribution is approximately

$f(c_i)\Delta x.$

Adding all those pieces gives a Riemann sum:

$\text{sum} f(c_i)\Delta x.$

Why the product matters

The product $f(c_i)\Delta x$ should be interpreted through units.

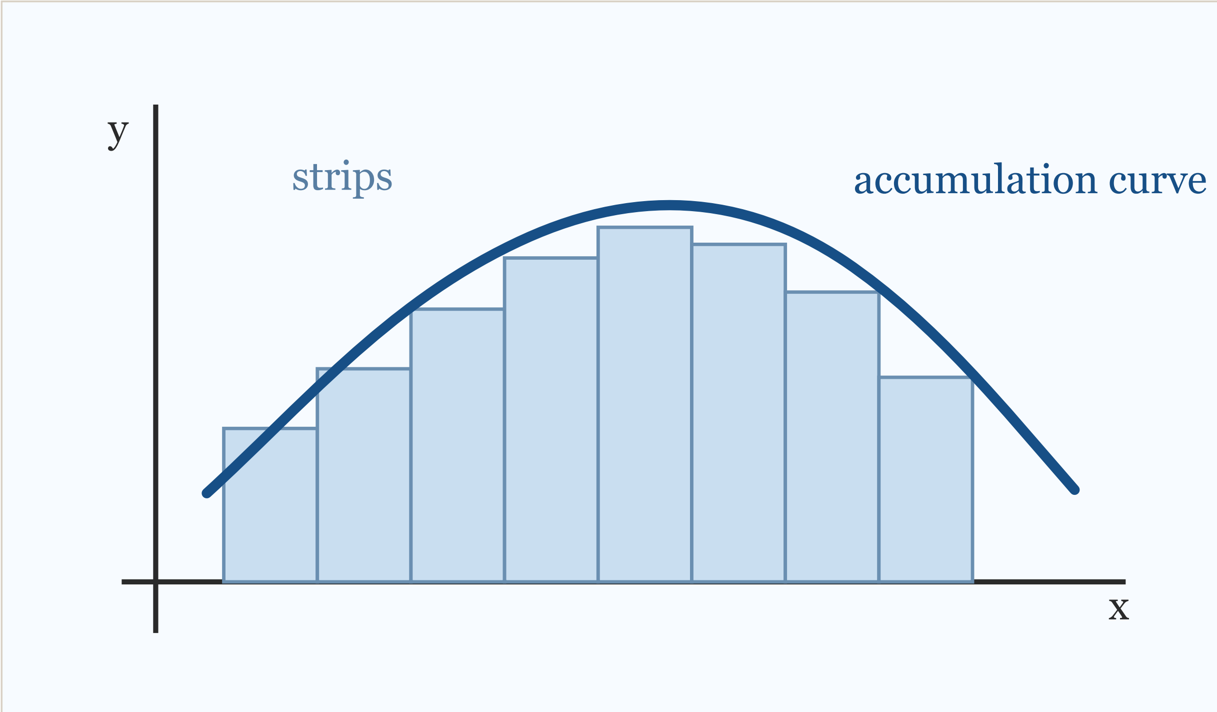
- If `f` is a rate in liters per minute and Δx is minutes, the product is liters.
- If `f` is a velocity in meters per second and Δx is seconds, the product is meters.
- If `f` is a density in kilograms per meter and Δx is meters, the product is kilograms.

The integral is powerful because the same structure works in many settings.

From sampled data to accumulated totals

Not every useful integral problem begins with a clean formula. Often we only know a rate at sampled times or measured positions. Riemann sums still apply. We estimate each small contribution from the data we have and add the pieces.

That is one reason calculus is not only a symbolic subject. It is also a numerical and modeling subject.



Example: left and right sums

Let $f(x) = x^2$ on `[0, 2]`, and split the interval into two equal pieces.

Then $\Delta x = 1$.

For a left-endpoint sum:

- first sample point: `0`
- second sample point: `1`

So the approximation is

$f(0)(1) + f(1)(1) = 0 + 1 = 1$.

For a right-endpoint sum:

- sample points: `1` and `2`

So the approximation is

$f(1)(1) + f(2)(1) = 1 + 4 = 5$.

The exact accumulated amount lies somewhere between these because the function is increasing.

Refining the partition

If we use more pieces, each piece becomes thinner and the approximation improves. The definite integral is the limit of this refinement process.

6.3 The definite integral

The definite integral of `f` from `a` to `b` is written

$\int_a^b f(x)dx.$

It is defined as the limit of Riemann sums, when that limit exists.

In words:

> The definite integral is the exact accumulated total obtained from finer and finer additive approximations.

Example: area under a line

Let $f(x) = x$ on `[0, 2]`.

Geometrically, the region under the graph is a right triangle with base `2` and height `2`, so its area is

$(1/2)(2)(2) = 2.$

That triangle formula comes from the geometric area function rather than from integration itself. Two congruent copies of the triangle form a parallelogram, and finite additivity reduces the problem to rectangle area.

Thus

$\int_0^2 x\,dx = 2.$

This example is geometric, but the same integral could be read in other ways:

- accumulated cost,
- total distance from a velocity graph,
- total charge from a current graph,
- and more.

Definite integral versus antiderivative

A definite integral is not merely an instruction to find an antiderivative. It is first a quantity.

An antiderivative becomes a computational tool later, but the meaning of the definite integral begins with accumulation.

That distinction matters because students often learn to compute before they know what is being computed.

Quantity first, shortcut second

This chapter deliberately reverses a common classroom habit. Instead of introducing the integral first as an inverse-derivative procedure, it introduces the integral as an accumulated quantity. That order is not cosmetic. It prevents an important misunderstanding.

If students learn only the shortcut, then every integral looks like a hunt for a formula. If they learn the quantity first, they can still reason when there is no convenient antiderivative, when the data are numerical, or when the context matters more than symbolic form.

6.4 Properties of accumulation

The definite integral behaves like a well-structured total.

A bookkeeping point that matters later

The additivity rule for integrals is more than a formal property. It is what allows large accumulation problems to be broken into intervals where the interpretation is clearer. In applications, one part of the interval may represent growth, another decay, another steady-state behavior. The integral keeps those pieces organized and recombinable.

Additivity over intervals

If $a < c < b$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

This matches common sense: total accumulation over a long interval equals the sum of the accumulations over the pieces.

Reversing the limits

If we reverse the direction of accumulation, the sign flips:

$$\int_b^a f(x)dx = -\int_a^b f(x)dx.$$

Integral of a constant

If $f(x) = k$, then

$\int_a^b k dx = k(b - a)$.

This agrees with the rectangle formula.

Comparison idea

If $f(x) \geq g(x)$ on `[a, b]`, then

$\int_a^b f(x) dx \geq \int_a^b g(x) dx$.

Bigger slices create a bigger total, provided the interval is the same.

Example: splitting net change

Suppose a population changes at rate $r(t)$ over a year. If you want total change from January to December, you can add:

- January to March,
- March to June,
- June to September,
- September to December.

The definite integral respects that bookkeeping exactly.

6.5 Average value over an interval

The average value of a continuous function `f` on `[a, b]` is

$(1/(b - a)) \int_a^b f(x) dx$.

This formula says:

1. accumulate the whole amount,
2. then spread it evenly across the interval length.

Why this is a genuine average

For a finite list of numbers, average means total divided by count.

For a continuously varying quantity, the integral provides the total, and the interval length plays the role of count. The result is a representative constant value that would produce the same total accumulation across the interval.

Example: average temperature model

Suppose a temperature model over the interval `[0, 4]` hours is

$T(t) = 20 + t.$

Then

$\int_0^4 (20 + t)dt = \int_0^4 20dt + \int_0^4 tdt = 80 + 8 = 88.$

So the average value is

$88/4 = 22.$

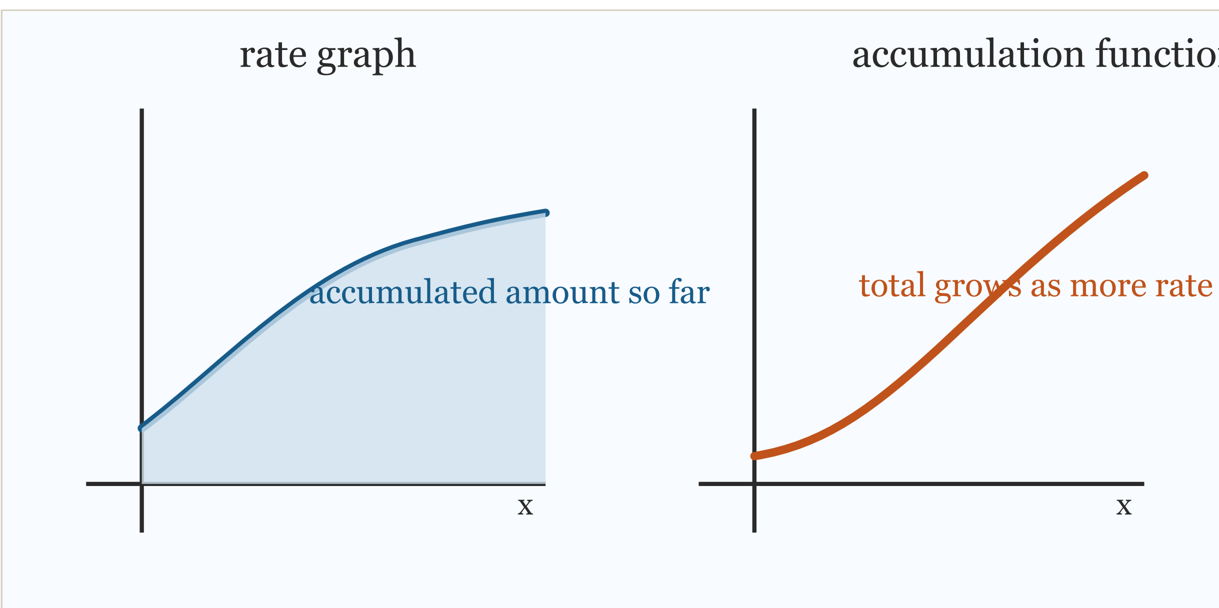
That means a constant temperature of `22` degrees over four hours would produce the same total accumulated temperature-hours as the changing model.

Average value is not always the midpoint of outputs

Students often guess that the average value on an interval is just the average of the endpoint function values. That sometimes works, but it is not the defining principle.

The true idea is equal-total replacement:

> The average value is the constant height that produces the same accumulated amount.



Quick tactics

- Name the increment being added before you write the integral. Expressions like `(rate)(time)` or `(density)(width)` prevent many setup mistakes.
- Decide early whether the problem wants net change, total amount, or average value. Those are related but not interchangeable.
- If the graph dips below the axis, say out loud what the negative contribution means in the context.
- If the data come from a table, do not wait for a perfect formula before estimating accumulation. Sums built from data are already meaningful calculus.

Common traps

- Treating the definite integral as only an area formula.
- Forgetting that regions below the axis count negatively in net change.
- Ignoring units in the product $f(x)\Delta x$.
- Confusing displacement with total distance.
- Treating average value as a purely algebraic recipe instead of an equal-total idea.

Proof window: why the constant-function rule is right

If $f(x) = k$, then each rectangle in a Riemann sum has height `k`. No matter how the interval is partitioned, the sum is

$$k(\Delta x_1 + \Delta x_2 + \ldots + \Delta x_n) = k(b - a).$$

Because every Riemann sum already has the same value, the limit is also $k(b - a)$.

This matters because it shows the integral definition agrees with simple geometric formulas in the cases where geometry is already familiar.

Exercises

Warm-up: integral meaning and units

1. What does $\int_a^b f(x)dx$ measure, in plain language?
2. If a velocity graph is below the axis, how does that affect displacement?
3. What is the average value of a constant function $f(x) = 7$ on any interval?

Core skill: Riemann sums and basic definite integrals

1. Compute the net change of a rate that is `4` for three hours and `-2` for one hour.
2. Write a left-endpoint Riemann sum with `3` equal subintervals for $f(x)$ on `[0, 6]`.
3. Find $\int_0^3 2dx$.
4. Use geometry to find $\int_0^4 x\,dx$.
5. Find the average value of $f(x) = 6$ on `[1, 5]`.

Interpretation: signed accumulation and average value

1. Explain why the units of $f(x)dx$ matter when interpreting an integral.
2. Describe a situation where net change and total amount traveled are different.

Challenge: net-change subtleties

1. Give an example of a function whose definite integral over an interval is `0` even though the function is not identically zero.
2. A function is positive on most of an interval but has negative values near the end. Explain how its definite integral could still be negative.
3. Explain in words why refining a Riemann sum should improve an accumulation estimate.

Modeling: rate-to-total problems

1. A pump fills a tank at rate $r(t) = 5 + t$ liters per minute for $0 \leq t \leq 4$. Interpret $\int_0^4 r(t)dt$.
2. A car's velocity is given in meters per second over a 10-second interval. Describe how you would use a Riemann sum to estimate displacement from sampled data.

Reflection

Integration begins with a simple idea: when a quantity changes by small pieces, totals are built by adding those pieces. The definite integral is calculus' disciplined version of that idea. It turns local rates into global totals.

Chapter 7. Antiderivatives and the Fundamental Theorem

Opening question

Suppose a runner's velocity at time `t` is known, but the position function is not. If you know where the runner started, can you recover where the runner is later?

This chapter answers that question. Differentiation describes local change. Integration builds totals. The Fundamental Theorem of Calculus connects the two.

Learning goals

By the end of this chapter, you should be able to:

- interpret an antiderivative as a family of functions,
- solve basic initial value problems,
- explain the Fundamental Theorem of Calculus in both of its main forms,
- differentiate area functions,
- and use substitution as a way of matching derivative structure.

Preview questions

- If two functions have the same derivative, how different can those functions be?
- Why does an integral with a variable upper limit behave like a new function rather than a single number?
- What clues tell you that substitution is the right antidifferentiation tool?

7.1 Reversing differentiation

If $F'(x) = f(x)$, then `F` is called an antiderivative of `f`.

For example:

- an antiderivative of `2x` is x^2,
- an antiderivative of $\cos x$ is $\sin x$,

- an antiderivative of e^x is e^x.

Antiderivatives are not unique. If $F'(x) = f(x)$, then $(F(x) + C)' = f(x)$ for any constant `C`.

That is why indefinite integrals are written with a constant:

$$intf(x)dx = F(x) + C.$$

Example: a simple antiderivative

Find $int(6x^2 - 4x + 3)dx$.

Work term by term:

- $int6x^2dx = 2x^3$
- $int - 4xdx = -2x^2$
- $int3dx = 3x$

So

$$int(6x^2 - 4x + 3)dx = 2x^3 - 2x^2 + 3x + C.$$

Antiderivative language matters

An indefinite integral is not a number. It is a family of functions. This is one of the first places in calculus where students often confuse an operation that returns a function with one that returns a quantity.

A reconstruction viewpoint

It helps to say explicitly what antiderivatives do. They reconstruct a changing quantity from knowledge of its rate.

If we know that a chemical concentration changes at rate `c'(t)`, then any antiderivative of `c'(t)` gives a possible concentration function. If we also know the initial concentration, then only one member of that family fits the situation.

This reconstruction viewpoint is why antiderivatives appear in motion, economics, population change, and heat transfer. They are not only algebraic inverses of derivatives. They are the mechanism by which local change becomes a full model over time.

Worked example cluster: constants and families

Find an antiderivative of each function and describe the role of the constant.

1. $f(x) = 8$
2. $g(x) = 3x^2 - 2x$
3. $h(x) = e^x + \cos x$

The antiderivatives are:

- $int8dx = 8x + C$
- $int(3x^2 - 2x)dx = x^3 - x^2 + C$
- $int(e^x + \cos x)dx = e^x + \sin x + C$

The recurring lesson is not the algebra. It is that a derivative loses constant information, so antidifferentiation must restore a family rather than a single graph.

7.2 Initial value problems

An initial value problem gives a differential relationship plus a starting condition.

Example: recover the function from its derivative

Suppose

$$y = 4x - 1$$

and

$y(2) = 7.$

First find the general antiderivative:

$y = 2x^2 - x + C.$

Now use the initial condition:

$7 = 2(2)^2 - 2 + C = 8 - 2 + C = 6 + C.$

So $C = 1$, and the solution is

$y = 2x^2 - x + 1.$

Why initial conditions matter

The derivative tells you the shape of change but not the vertical placement of the graph. The initial condition fixes the correct member of the antiderivative family.

Motion interpretation

If velocity is known and initial position is known, antiderivatives recover position. If acceleration is known and initial velocity is known, antiderivatives recover velocity.

This is why antiderivatives are not only symbolic inverses. They are reconstruction tools.

Example: recovering position from velocity data

Suppose a particle moves with velocity

$$v(t) = 6t - 4$$

and starts at position $s(0) = 3$.

An antiderivative of `v` is

$s(t) = 3t^2 - 4t + C.$

Using the initial condition,

$3 = s(0) = C.$

So

$s(t) = 3t^2 - 4t + 3.$

This is a simple example, but it captures a major modeling principle: rate information alone describes a family; initial data selects the physically relevant member.

7.3 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus has two complementary forms.

Part I: differentiating an accumulation function

Define

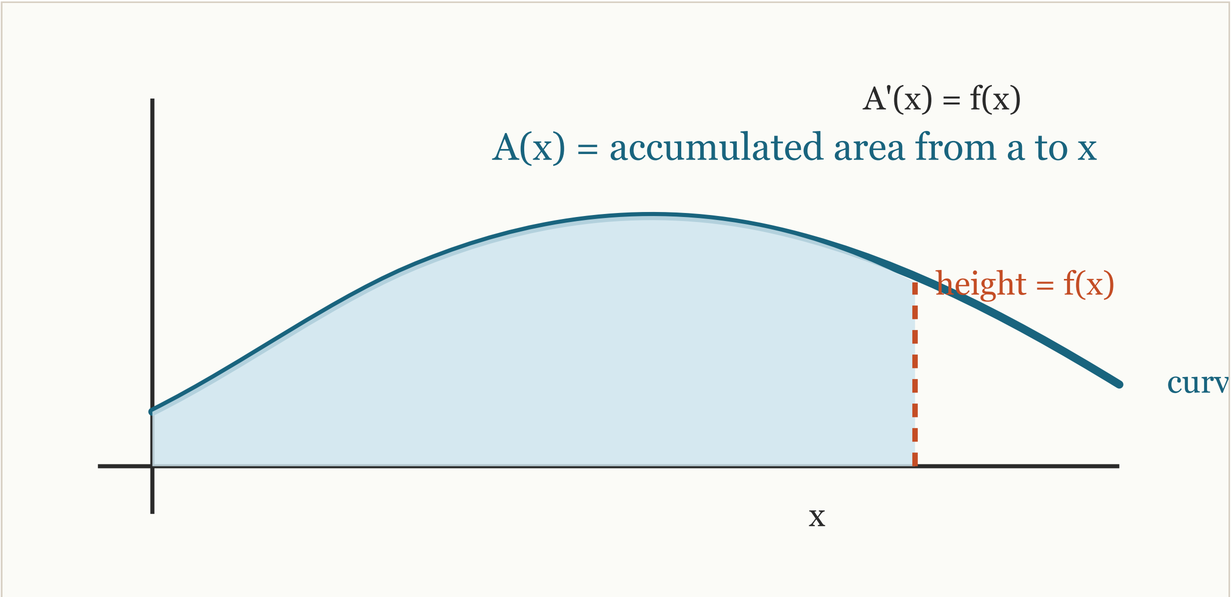
$A(x) = \int_a^x f(t)dt.$

If `f` is continuous on an interval containing `x`, then

$A'(x) = f(x).$

This says:

> If you build a function by accumulating a rate, then differentiating that total returns the original rate.



This is not a trick. It is a structural statement about change and accumulation.

Part II: computing definite integrals

If `f` is continuous on `[a, b]` and `F` is any antiderivative of `f` on `[a, b]`, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

This tells us how to compute many definite integrals without taking limits of Riemann sums by hand.

Example: computing a definite integral

Find

$$\int_1^3 2x\,dx.$$

An antiderivative of `2x` is x^2, so

$$\int_1^3 2x\,dx = 3^2 - 1^2 = 9 - 1 = 8.$$

Why this matters

Before the theorem, the definite integral was defined through accumulation and limiting sums. After the theorem, it becomes computable by antiderivatives. Neither viewpoint should replace the other:

- the integral means accumulation,
- the antiderivative method is the shortcut made possible by the theorem.

Local-to-global and global-to-local

The two directions of the Fundamental Theorem are often memorized separately, but they are easier to retain when described in a pair:

- Part I says that if you accumulate a rate and then differentiate the accumulation, you recover the rate.
- Part II says that if you know a function whose derivative is the rate, then total accumulation over an interval is found by endpoint comparison.

One direction turns accumulation into a derivative. The other turns a definite integral into endpoint evaluation. Together they explain why calculus can move back and forth between local and global information.

A quantity-based reading

In classroom practice, many students can manipulate symbols correctly but still misread what the theorem means. A good repair is to insert units into the discussion.

If $r(t)$ is measured in liters per minute, then

$$\int_0^5 r(t)dt$$

has units of liters. The integral is a total amount, not a rate. If we define

$R(x) = \int_0^x r(t)dt,$

then `R'(x)` has units of liters per minute again because differentiation converts the accumulation function back into an instantaneous rate.

The units help explain the theorem without any mystery.

7.4 Area functions

An area function is a function defined by an integral with a variable upper limit.

For example,

$G(x) = \int_0^x t^2dt.$

This function does not return a slope or an algebraic expression at first glance. It returns a total accumulated quantity from `0` to `x`.

By the Fundamental Theorem,

$$G'(x) = x^2.$$

Example: differentiate a shifted area function

Let

$$H(x) = \int_2^x (t^3 + 1)dt.$$

Then

$$H'(x) = x^3 + 1.$$

The derivative is read directly from the integrand, with the variable `t` replaced by `x`.

If the upper limit is not just `x`

Suppose

$$K(x) = \int_0^{x^2} \cos t\,dt.$$

Then the upper limit is itself changing with `x`, so the chain rule must join the Fundamental Theorem:

$$K'(x) = \cos(x^2)(2x).$$

This is a useful reminder that calculus ideas usually connect rather than live alone.

Geometry of the constant lower limit

In an area function such as

$$G(x) = \int_2^x f(t)dt,$$

the lower limit `2` is not decorative. It fixes the reference point from which accumulation is measured. If you changed the lower limit to `0`, you would obtain a different function, but its derivative would still be $f(x)$ as long as the upper limit remains `x`.

This is a subtle but important idea:

- changing the lower limit changes the antiderivative by a constant,
- and differentiation ignores that constant difference.

That observation quietly unifies Part I of the Fundamental Theorem with the earlier language of antiderivative families.

7.5 Substitution as structure recognition

Substitution is the antidifferentiation partner of the chain rule.

If a function has the form

$f(g(x))g'(x)$,

then substitution often simplifies the problem.

Example: an indefinite integral

Find

$int 2x\cos(x^2)dx$.

Let

$u = x^2$,

so

$du = 2x dx$.

Then the integral becomes

$int \cos(u)du = \sin(u) + C = \sin(x^2) + C$.

Example: a definite integral

Find

$\int_0^1 2xe^{x^2}dx$.

Let $u = x^2$, so $du = 2x dx$. The limits also change:

- when $x = 0, u = 0$
- when $x = 1, u = 1$

So

$\int_0^1 2xe^{x^2}dx = \int_0^1 e^u du = e - 1$.

Substitution is not guessing

Good substitution depends on structure recognition:

- find an inside expression,
- check whether its derivative is present up to a constant factor,
- and rewrite the integral in the simpler variable.

This is one of the first places where mature calculus starts to feel strategic.

A substitution checklist

Before starting a substitution, ask:

1. Is there a natural inside expression?
2. Does its derivative appear exactly or up to a constant multiple?
3. Will the new variable actually make the integrand simpler?

If the answer to the third question is no, then the substitution is cosmetic rather than helpful.

For example, in

$$\mathrm{int} x/(x^2 + 9)dx,$$

the denominator $x^2 + 9$ is the clear inside expression, and its derivative is `2x`, which appears up to a constant factor. That is a strong signal for substitution. In contrast, for

$$\mathrm{int} x^2 e^x dx,$$

no natural inside derivative is present, so substitution is usually weaker than integration by parts.

Quick tactics

- Read $\mathrm{int} f(x)dx$ and $\int_a^b f(x)dx$ as different kinds of objects from the start.
- When solving an initial value problem, delay simplification until after the constant has been found.
- For area functions, differentiate the integrand first in your head, then check whether a chain rule factor is needed.
- Use units to test whether an integral should represent an amount, an average, or a rate.

Chapter review

This chapter is the hinge of single-variable calculus. Derivatives describe local change. Integrals describe accumulated totals. The Fundamental Theorem explains why these two ideas are not separate topics but two faces of one structure.

A reader who truly understands this chapter should be able to do more than compute. They should be able to answer questions like:

- What information is lost when we differentiate?
- What information is restored when we antidifferentiate?
- Why does an accumulation function produce a new function whose derivative is the current rate?
- Why does substitution mirror the chain rule?

Those questions point toward long-term understanding, not just short-term exam performance.

Mini projects

Project 1: accumulation diary

Choose a rate process from everyday life: water entering a tank, distance traveled over time, money deposited into an account, or electricity consumed during a day. Build a small table of rate values over time, sketch the rate graph, explain what the definite integral means in context, and describe the units carefully.

Project 2: compare two antiderivative viewpoints

Write a short essay comparing the "family of functions" viewpoint for indefinite integrals with the "accumulated amount" viewpoint for definite integrals. Include at least three examples where mixing those viewpoints causes mistakes.

Common traps

- Forgetting the constant `C` in an indefinite integral.
- Treating a definite integral like an indefinite one and adding `C`.
- Using the Fundamental Theorem mechanically without remembering the quantity meaning.
- Forgetting to change limits in a definite-integral substitution.
- Trying substitution when the derivative structure is not actually present.

Proof window: why $d/dx \int_a^x f(t)dt = f(x)$ makes sense

If $A(x) = \int_a^x f(t)dt$, then changing `x` to `x + h` adds only a small extra slice:

$A(x + h) - A(x) = \int_x^{x+h} f(t)dt.$

Over that short interval, the accumulated amount is approximately

$f(x)h.$

Dividing by `h` gives something close to $f(x)$, and in the limit the approximation becomes exact under continuity assumptions.

The key idea is local:

> The rate of change of an accumulated total is the current rate being accumulated.

Exercises

Warm-up: antiderivatives and FTC meaning

1. What is an antiderivative of `6x`?
2. Why does an indefinite integral include `+ C`?
3. State the computational form of the Fundamental Theorem of Calculus.

Core skill: antiderivatives and FTC computation

1. Find $int(3x^2 - 8)dx.$
2. Solve $y' = 5x^4$ with $y(1) = 3.$
3. Compute $\int_0^2 3x^2 dx.$
4. Differentiate $G(x) = \int_1^x (t^2 + 4)dt.$
5. Compute $\int_0^2 2xe^{x^2} dx.$

Interpretation: derivatives and integrals together

1. Explain in words how the Fundamental Theorem links derivatives and integrals.
2. Explain why $\int_a^b f(x)dx$ and $intf(x)dx$ are different kinds of objects.

Challenge: accumulation inside compositions

1. Let $F(x) = \int_0^{x^3} \sin t dt$. Find `F'(x)`.
2. Give an example of two different antiderivatives of the same function.

3. Explain how an initial value problem chooses one function from an antiderivative family.

Modeling: motion and reservoir totals

1. The acceleration of a car is $a(t) = 2t$ meters per second squared, and $v(0) = 5$. Find the velocity function.
2. A reservoir gains water at rate $r(t) = 4 + 0.5t$ cubic meters per hour for $0 \le t \le 6$. Interpret and compute $\int_0^6 r(t)dt$.

Reflection

The Fundamental Theorem of Calculus is not merely a theorem to memorize. It is the hinge of the subject. It says that a local rate and a global accumulation are two sides of one structure.

Chapter 8. More Integration Tools

Opening question

Not every integral announces its method. Some integrals match a known derivative pattern immediately. Others need rewriting, decomposition, or numerical approximation.

This chapter is about method choice. Integration is not only about rules. It is about recognizing structure.

Learning goals

By the end of this chapter, you should be able to:

- use integration by parts,
- recognize common trigonometric integration patterns,
- use trigonometric substitution in geometric settings,
- decompose basic rational functions with partial fractions,
- and apply numerical integration when an elementary antiderivative is unavailable or impractical.

Preview questions

- When should a product trigger substitution, and when should it trigger integration by parts?
- Why do some radical expressions naturally suggest a trigonometric substitution?
- How do you make a principled choice when an exact elementary antiderivative is unlikely?

8.1 Integration by parts

Integration by parts comes from the product rule:

$d/dx(uv) = uv' + vu'$.

Rearranging and integrating gives

$$intudv = uv - intvdu.$$

Choosing `u` and `dv`

The goal is not random substitution. It is simplification. Choose:

- `u` to become simpler when differentiated,
- `dv` to be easy to integrate.

Example: xe^x

Find

$$int \times e^x dx.$$

Choose

- $u = x$, so $du = dx$
- $dv = e^x dx$, so $v = e^x$

Then

$$int \times e^x dx = xe^x - inte^x dx = xe^x - e^x + C.$$

So

$$int \times e^x dx = e^x (x - 1) + C.$$

Example: $x \cos x$

Choose

- $u = x$
- $dv = \cos \times dx$

Then

$$int \times \cos \times dx = x \sin x - int \sin \times dx = x \sin x + \cos x + C.$$

Integration by parts is especially useful when algebraic and exponential, logarithmic, or trigonometric factors appear together.

A practical heuristic for choosing `u`

In many courses, students learn an ordering rule for choosing `u` in integration by parts. The deeper idea is simpler: differentiate the factor that gets simpler, integrate the factor that stays manageable.

For instance:

- xe^x suggests differentiating `x`,
- $x \sin x$ suggests differentiating `x`,
- $\ln x$ suggests differentiating $\ln x$,
- but $e^x \cos x$ does not simplify much under either choice, so the method becomes more involved.

There is no universal ranking that replaces judgment. Large calculus texts become dense partly because they include enough varied examples for that judgment to become stable.

Example: $int \ln \times dx$

This integral looks like it lacks the second factor required by integration by parts, but we can write

$int \ln \times dx = int(\ln x)(1)dx.$

Choose

- $u = \ln x$, so $du = dx/x$
- $dv = dx$, so $v = x$

Then

$int \ln \times dx = x \ln x - int x(1/x)dx = x \ln x - int 1dx.$

Therefore

$int \ln \times dx = x \ln x - x + C.$

This standard example teaches a useful habit: when a method seems unavailable, rewrite the problem until the hidden structure becomes visible.

8.2 Trigonometric integrals

Trigonometric integrals often reward pattern recognition more than brute force.

Example: powers of sine and cosine

Find

$int \sin \times \cos \times dx.$

Let

$u = \sin x,$

so

$du = \cos \times dx.$

Then

$int \sin \times \cos \times dx = intudu = (1/2)u^2 + C = (1/2)\sin^2 x + C.$

Example: secant squared

Because

$d/dx(\tan x) = \sec^2 x,$

we have

$int \sec^2 \times dx = \tan x + C.$

Strategy notes

Common useful ideas include:

- separating one factor to create a substitution,
- using identities such as $\sin^2 x + \cos^2 x = 1$,
- and recognizing standard derivative patterns quickly.

At first, this feels like memory work. With practice, it becomes structural reading.

Identity bank for common trig patterns

Three identity moves occur so often that they deserve to stay close at hand:

- $\sin^2 x + \cos^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \csc^2 x$

These identities help in two ways. Sometimes they simplify the integrand directly. Other times they allow one factor to be saved for substitution while the remaining power is rewritten in a friendlier form.

For example, in

$int \sin^3 \times \cos \times dx,$

the most natural move is to let $u = \sin x$, since $du = \cos \times dx$ is already present. In more complicated examples, the identity step and the substitution step work together rather than separately.

8.3 Trigonometric substitution

Trigonometric substitution is useful when radicals such as

- $\sqrt{a^2 - x^2}$,
- $\sqrt{a^2 + x^2}$,
- $\sqrt{x^2 - a^2}$

appear.

The point is geometric: trigonometric identities can replace a difficult radical with a cleaner expression.

Example: $\sqrt{4 - x^2}$

Consider

$int\sqrt{4 - x^2}dx.$

Let

$x = 2 \sin \theta.$

Then

$$dx = 2 \cos \theta dtheta$$

and

$\sqrt{4 - x^2} = \sqrt{4 - 4 \sin^2 \theta} = 2 \cos \theta.$

So the integral becomes

$int(2 \cos \theta)(2 \cos \theta)dtheta = 4int \cos^2 \theta dtheta.$

The calculation continues with trigonometric identities.

This technique is longer than substitution in Chapter 7, but the same principle applies: rewrite the integral into a friendlier form.

When to use it

Trigonometric substitution is not the first method to try. It is usually chosen when the radical structure strongly suggests it.

Geometry behind the substitution

The three classic radical patterns correspond to familiar trigonometric identities:

- $a^2 - x^2$ matches $1 - \sin^2\theta = \cos^2\theta$,
- $a^2 + x^2$ matches $1 + \tan^2\theta = \sec^2\theta$,
- $x^2 - a^2$ matches $\sec^2\theta - 1 = \tan^2\theta$.

The method feels less arbitrary once you notice that the radical is being converted into a single trigonometric factor by an identity. That geometric motivation is why many print textbooks pair this algebra with a right-triangle sketch.

8.4 Partial fractions

Partial fractions break a rational function into simpler pieces that can be integrated separately.

Example: distinct linear factors

Find

$int 1/(x^2 - x - 2)dx.$

Factor the denominator:

$x^2 - x - 2 = (x - 2)(x + 1).$

Write

$1/((x - 2)(x + 1)) = A/(x - 2) + B/(x + 1).$

Then

$1 = A(x + 1) + B(x - 2).$

Set convenient values:

- if $x = 2$, then $1 = 3A$, so $A = 1/3$
- if $x = -1$, then $1 = -3B$, so $B = -1/3$

So

$$\int 1/(x^2 - x - 2)dx = (1/3)\int 1/(x - 2)dx - (1/3)\int 1/(x + 1)dx.$$

Therefore

$$\int 1/(x^2 - x - 2)dx = (1/3)\ln|x - 2| - (1/3)\ln|x + 1| + C.$$

Why this works

Partial fractions turns a complicated rational expression into a sum of standard antiderivative forms.

Beyond distinct linear factors

The simplest partial fractions problems involve distinct linear factors, but the method extends further:

- repeated linear factors produce several terms, such as $A/(x - 1) + B/(x - 1)^2$,
- irreducible quadratic factors produce linear numerators, such as $(Ax + B)/(x^2 + 1)$.

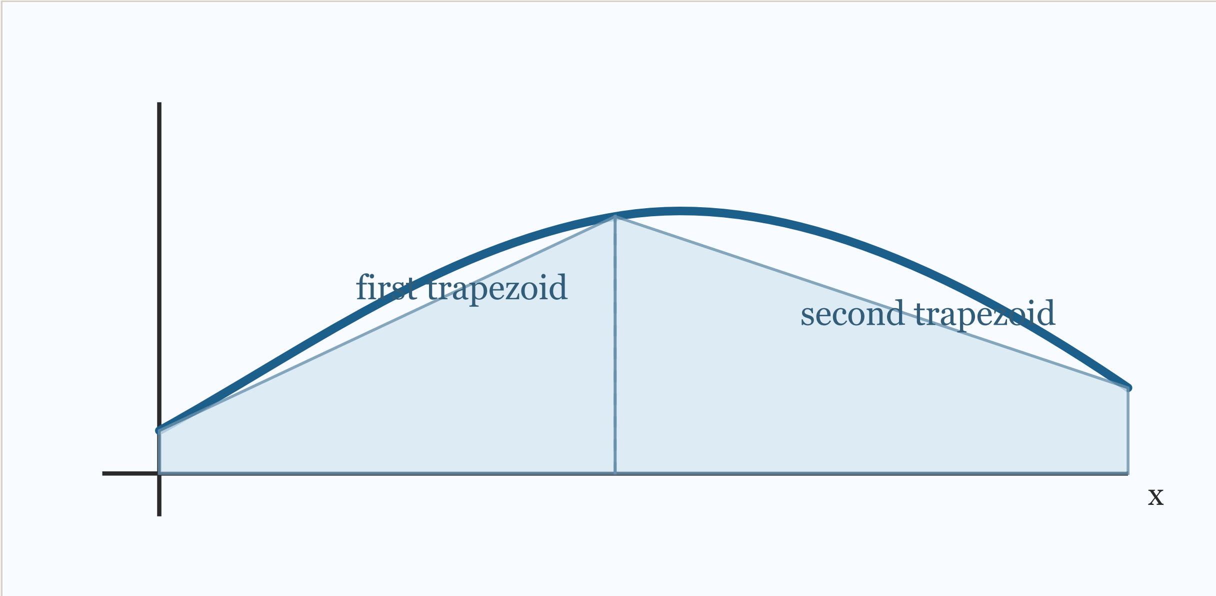
The general principle does not change. Match the algebraic structure of the denominator, solve for the coefficients, and reduce the integral to pieces whose antiderivatives are already known.

8.5 Numerical integration

Some definite integrals are hard or impossible to compute with elementary antiderivatives. Numerical methods estimate the value from sampled data or function values.

The trapezoidal rule

The trapezoidal rule approximates the area under the curve by trapezoids instead of rectangles.



On one interval `[a, b]`, the estimate is

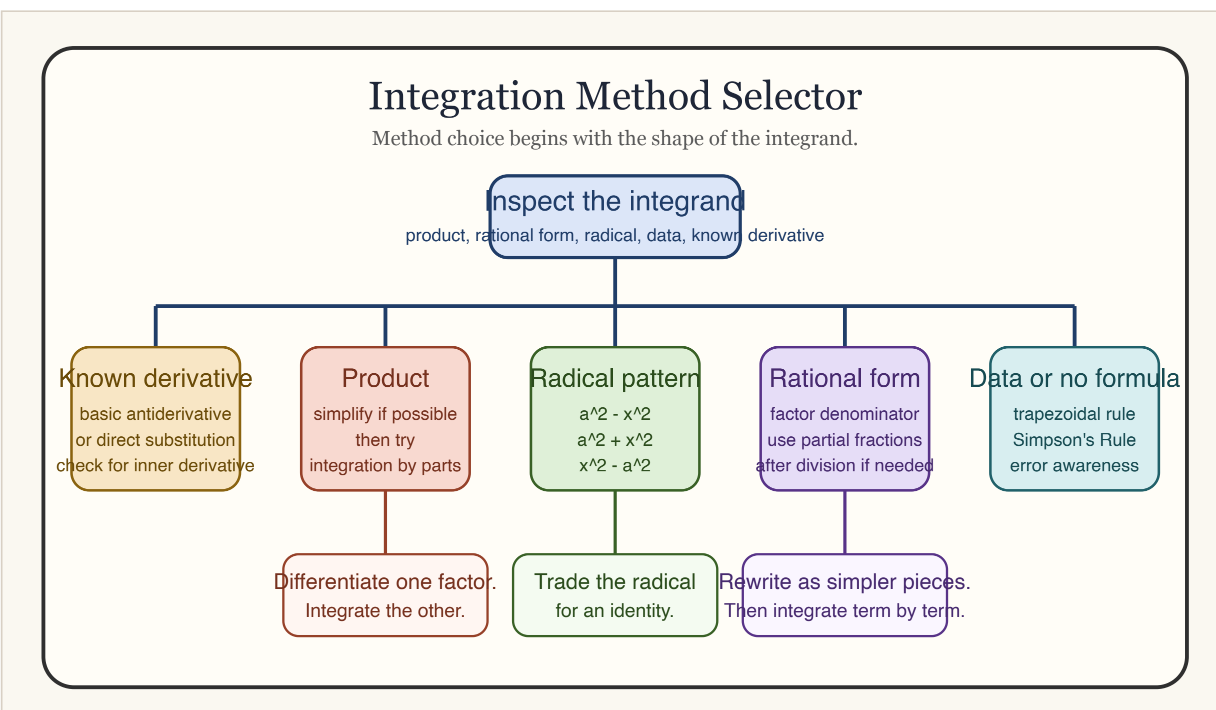
$(b - a)(f(a) + f(b))/2.$

With more subintervals, the rule becomes more accurate for many smooth functions.

Simpson's Rule

Simpson's Rule uses parabolic arcs and often improves accuracy for smooth functions. The main idea is the same: replace the actual curve by something easier to integrate exactly on short intervals.

A method-selection map



Example: trapezoidal estimate

Estimate

$$\int_0^2 x^2 dx$$

using two trapezoids.

Use the points:

- $x = 0, f(x) = 0$
- $x = 1, f(x) = 1$
- $x = 2, f(x) = 4$

Each subinterval has width `1`. The trapezoidal estimate is

$(1/2)(0 + 2(1) + 4) = 3.$

The exact integral is $8/3$, so the estimate is close but not exact.

A fuller trapezoidal-rule example

Estimate

$$\int_0^4 f(x) dx$$

from the table

`x`	`0`	`1`	`2`	`3`	`4`
$f(x)$	`3.0`	`3.8`	`4.1`	`4.7`	`5.0`

With step size $h = 1$, the trapezoidal estimate is

$T = (1/2)[3.0 + 2(3.8 + 4.1 + 4.7) + 5.0].$

So

$T = (1/2)[3.0 + 7.6 + 8.2 + 9.4 + 5.0] = (1/2)(33.2) = 16.6.$

This example matters because it resembles actual experimental data. In many scientific settings there is no closed formula to integrate, only measurements. Numerical integration is therefore part of ordinary applied practice, not a fallback reserved for unusual problems.

Error awareness for numerical methods

The trapezoidal rule often overestimates when the graph is concave up and underestimates when the graph is concave down, though the exact behavior depends on the partition. Simpson's Rule typically performs better on smooth curves because quadratic pieces can bend in a way trapezoids cannot.

When numerical integration matters

Numerical methods matter when:

- the antiderivative is unavailable,
- data are sampled experimentally,
- or speed matters more than symbolic exactness.

Modern calculus should not pretend that every meaningful integral ends in a neat closed form.

Quick tactics

- Try `substitution` when one part of the integrand looks like the derivative of another part.
- Try `integration by parts` when the integrand is a product and differentiating one factor will simplify it.
- Try `partial fractions` when the integrand is rational and the denominator factors cleanly.
- Use `numerical integration` when the input is experimental data or when exact symbolic work is not the main goal.
- If two methods seem possible, choose the one that most clearly reduces complexity after one step.

Chapter review

This chapter is about decision-making under uncertainty. The formulas matter, but the larger skill is diagnostic:

- identify the algebraic or geometric structure,
- predict what a successful rewrite would look like,
- choose the method that creates that rewrite,
- and differentiate the result afterward to confirm the answer.

A productive way to study Chapter 8 is to group problems by structural cue rather than by book order. Put together product integrals, rational integrals, radical integrals, and data-based integrals. The shared method signals become much easier to see.

Mini projects

Project 1: build an integration field guide

Collect thirty integrals from Chapters 7 and 8. For each one, record the chosen method, the structural clue that triggered that method, and one common wrong first move. The result should be a compact field guide to integration strategy.

Project 2: exact versus numerical

Choose three definite integrals that can be computed exactly and estimate each one with the trapezoidal rule and Simpson's Rule. Compare the errors and describe how the graph shape helps explain the differences.

Common traps

- Applying integration by parts in a way that makes the integral harder.
- Using identities incorrectly in trigonometric integrals.
- Starting trigonometric substitution when a simpler method would work.
- Forgetting absolute values in logarithmic antiderivatives from partial fractions.
- Treating numerical estimates as exact.

Proof window: why integration by parts is reasonable

Integration by parts is not a mysterious new formula. It is the product rule viewed backward. If differentiation of a product distributes change across both factors, then antidifferentiation can sometimes reverse that process by moving difficulty from one factor to the other.

That is why good `u` choice matters so much: the formula is only useful when the rearrangement simplifies the remaining integral.

Exercises

Warm-up: choosing an integration tool

1. State the integration by parts formula.
2. Why is $int\ \sec^2 \times dx$ easy to recognize?
3. What is the main purpose of numerical integration?

Core skill: parts, substitution, and fractions

1. Compute $int \times e^x dx$.
2. Compute $int \times \sin \times dx$.
3. Compute $int\ \sec^2 \times dx$.
4. Compute $int x/(x^2 + 1)dx$.
5. Decompose and integrate $int 1/(x^2 - 1)dx$.

Interpretation: method choice and approximation

1. Explain when substitution is better than integration by parts.
2. Explain why numerical integration belongs in a calculus book even if it does not always give exact answers.

Challenge: nonroutine integrals and error reasoning

1. Use integration by parts to compute $int\ \ln \times dx$.
2. Describe the denominator patterns that suggest partial fractions.
3. Explain how the trapezoidal rule improves on a rectangle rule for many curved graphs.

Modeling: data-driven and physical integrals

1. The power output of a machine is known at equally spaced times, but only as table data. Explain how the trapezoidal rule could estimate total energy output.
2. A physics problem produces an integral involving $\sqrt{9 - x^2}$. Explain why trigonometric substitution is a natural method to try.

Reflection

By now, integration should feel less like one operation and more like a family of strategies. The hard part is often not the algebra. It is seeing the shape of the problem clearly enough to choose the right method.

Chapter 9. Applications of Integration

Opening question

If a curve encloses a region, a solid, or a path, how can many thin slices combine into an exact total?

Applications of integration work because a difficult object can often be broken into simpler pieces whose small contributions are easy to understand.

Learning goals

By the end of this chapter, you should be able to:

- set up integrals for area between curves,
- use slicing and shells to compute volumes,
- interpret arc length and surface area formulas,
- and model work, mass, and other accumulated effects with integrals.

Preview questions

- How do you decide whether a geometric application should use vertical slices, horizontal slices, disks, washers, or shells?
- Why do formulas for arc length and surface area contain square roots?
- What common blueprint connects area, volume, work, mass, and fluid force?

Foundations note: volume is also a set function

Just as area is a function on a class of planar regions, volume is a function on a class of solids. For boxes, prisms, and polyhedra, the organizing properties are normalization on the unit cube, congruence invariance, and finite additivity.

Slicing methods extend that older geometric logic rather than replacing it. In particular, formulas such as $V(\text{box}) = abc$, $V(\text{prism}) = Bh$, and $V = 1/3Bh$ for pyramids and tetrahedra belong to the same family of additive measurement ideas.

Appendix S develops this point more explicitly, including where the $1/3$ factor comes from and why some bounded sets fail to have Jordan volume.

9.1 Area between curves

If one curve lies above another on an interval `[a, b]`, then the area between them is

$\int_a^b (\mathrm{top} - \mathrm{bottom})dx.$

Example: area between $y = x$ and $y = x^2$

The curves meet when

$x = x^2,$

so $x = 0$ and $x = 1$.

On `[0, 1]`, the line $y = x$ lies above the parabola $y = x^2$, so

$\mathrm{Area} = \int_0^1 (x - x^2)dx.$

Compute:

$\mathrm{Area} = [x^2/2 - x^3/3]_0^1 = 1/2 - 1/3 = 1/6.$

Why the subtraction works

Each thin vertical slice has height

`top - bottom`.

Integration adds those slice areas.

Horizontal slices versus vertical slices

The formula $\int_a^b (\mathrm{top} - \mathrm{bottom})dx$ is not the only possible setup. Sometimes a region is easier to describe with horizontal slices, in which case the area becomes

$\int_c^d (\mathrm{right} - \mathrm{left})dy.$

This choice matters because many area problems become artificially hard when the slice direction is chosen poorly. A useful habit is to ask which orientation makes the boundary functions easiest to describe.

9.2 Volumes by slices and shells

Volumes are built from cross-sectional areas or cylindrical shells.

Slicing

If a solid has cross-sectional area $A(x)$ perpendicular to the `x`-axis, then its volume is

$\int_a^b A(x)dx$.

Example: disks

Rotate $y = x$ on `[0, 2]` about the `x`-axis.

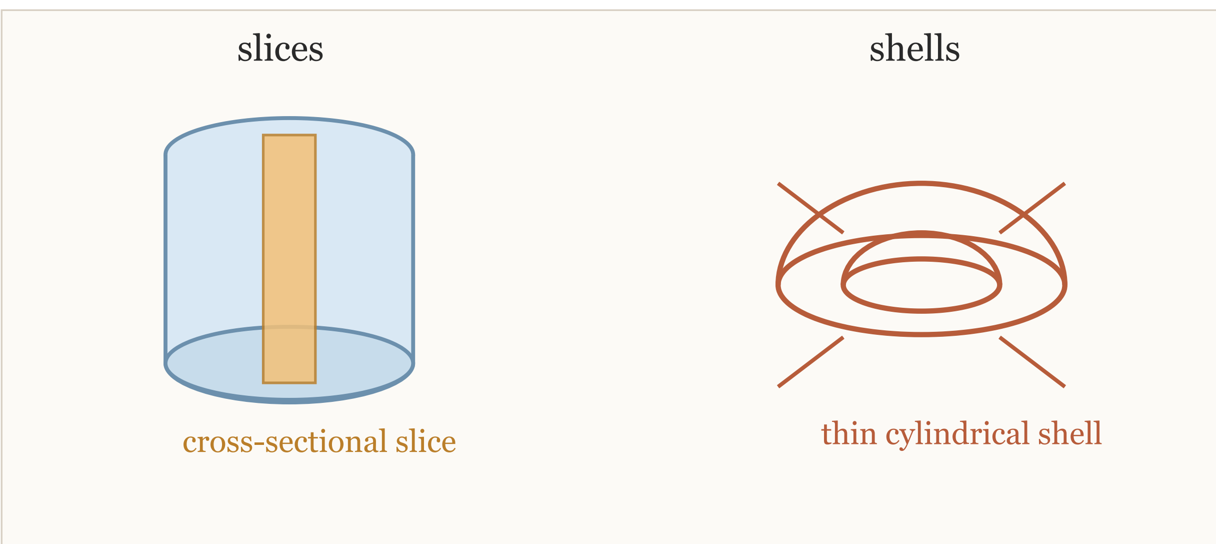
Each cross-section is a disk of radius `x`, so its area is πx^2. Therefore,

$V = \int_0^2 \pi x^2 dx = \pi[x^3/3]_0^2 = 8pi/3$.

The coefficient $1/3$ in this answer is not accidental. It echoes the classical pyramid and tetrahedron formula $V = 1/3Bh$, where a quadratic cross-sectional scaling law integrates to a cubic total.

Shells

Sometimes cylindrical shells are cleaner than disks.



For shells, the small volume is approximately

$2\pi(\text{radius})(\text{height})(\text{thickness})$.

The method chosen should reflect the geometry, not habit.

Example: shells for a rotated region

Rotate the region under $y = x$ on `[0, 1]` about the `y`-axis.

Using vertical shells:

- radius = `x`
- height = `x`
- thickness = dx

So

$$V = \int_0^1 2\pi x(x)dx = 2\pi \int_0^1 x^2 dx = 2pi/3.$$

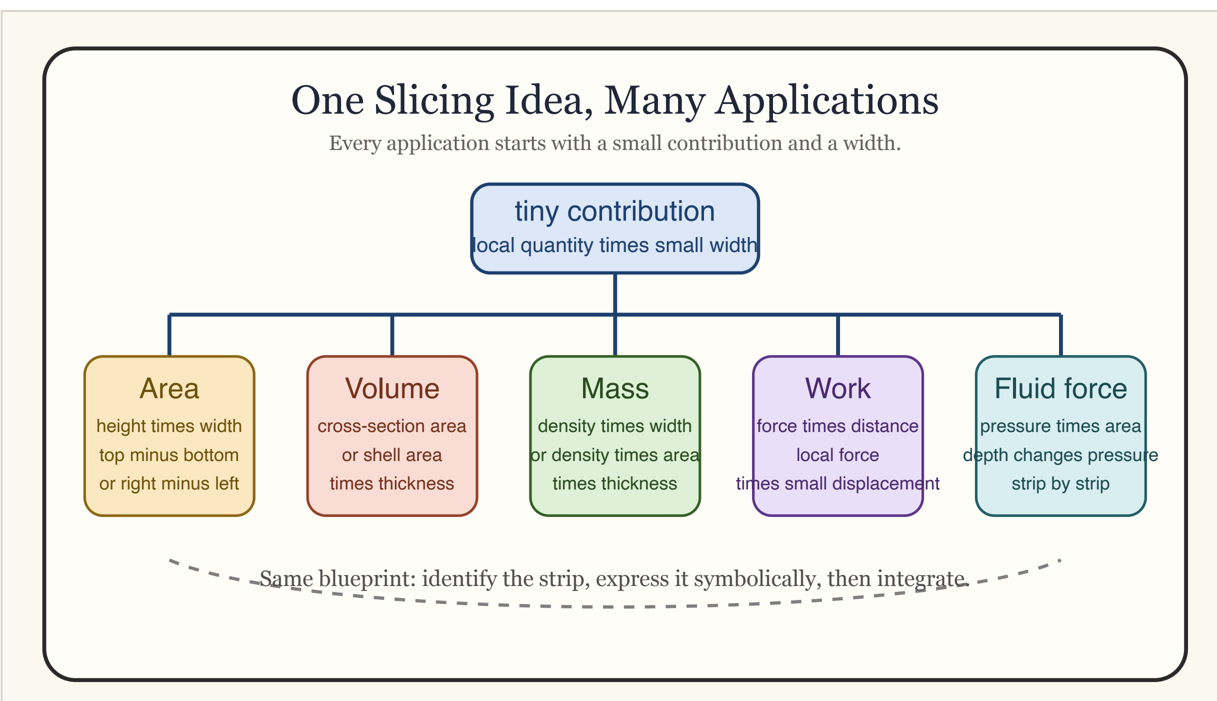
This example is simple on purpose. Its main lesson is strategic: the shell method often avoids solving for `x` as a function of `y`.

Washer-versus-shell checklist

When solids of revolution appear, use the following questions:

1. Are slices perpendicular or parallel to the axis of rotation?
2. If perpendicular, will the cross-sections be disks or washers?
3. If parallel, will shells produce a simpler radius-height description?
4. Which choice avoids unnecessary algebraic inversion?

Students often memorize disk and shell formulas separately, but both come from the same small-volume idea. The difference is geometric convenience, not a change in principle.



9.3 Arc length

A curve is not a stack of rectangles, but it can still be approximated by many short straight segments.

For a smooth graph $y = f(x)$ on `[a, b]`, the arc length is

$L = \int_a^b \mathrm{sqrt}(1 + (f'(x))^2) dx.$

Why this formula appears

On a tiny interval, the graph is almost a line segment. Using the Pythagorean Theorem,

$smalllength \approx \mathrm{sqrt}((\Delta x)^2 + (\Delta y)^2).$

Factor out Δx:

$smalllength \approx \mathrm{sqrt}(1 + (\Delta y / \Delta x)^2) \Delta x.$

In the limit, the slope $\Delta y / \Delta x$ becomes `f'(x)`.

Example: a line segment

If $f(x) = 3x$ on `[0, 2]`, then

$f'(x) = 3.$

So

$L = \int_0^2 \sqrt{1 + 9} dx = \int_0^2 \sqrt{10} dx = 2\sqrt{10}.$

That matches the direct distance formula.

Local straightness matters

Arc length works because smooth curves are locally almost straight. The square-root expression

$$\mathrm{sqrt}(1 + (f'(x))^2)$$

measures how much longer a tiny slanted piece of the curve is than its horizontal projection. When the slope is small, this factor stays close to `1`; when the slope is large, the curve stretches more rapidly.

9.4 Surface area

When a graph is revolved around an axis, its surface area can also be built from many thin bands.

For rotation about the `x`-axis, the surface area is

$$S = \int_a^b 2\pi f(x) \mathrm{sqrt}(1 + (f'(x))^2) dx,$$

provided $f(x) \geq 0$.

This formula combines:

- circumference of the band,
- tiny slanted length,
- and accumulation across the interval.

Geometric meaning

The factor $2\pi f(x)$ is a circumference. The square-root factor is the local stretching caused by the slant of the curve. The integral adds all the bands.

Example: surface area of a cone-like graph

Rotate $y = x$ on `[0, 1]` about the `x`-axis. Then

- $f(x) = x$
- $f'(x) = 1$

So

$$S = \int_0^1 2\pi x \sqrt{1 + 1^2} dx = 2\pi\sqrt{2} \int_0^1 \times dx.$$

Therefore

$$S = 2\pi\sqrt{2}[x^2/2]_0^1 = \pi\sqrt{2}.$$

This example shows that surface-area formulas are more sensitive than volume formulas to slope, because the slant factor remains in the integrand.

9.5 Work, mass, and accumulated effects

Integration applies whenever a total is built from small pieces.

Work

If a variable force $F(x)$ moves an object along a line, the work is

$$W = \int_a^b F(x)dx.$$

Mass from density

If a rod has linear density $\varrho(x)$, then its mass is

$$M = \int_a^b \varrho(x)dx.$$

Fluid force and pressure

When pressure changes with depth, the total force on a submerged plate is built from many small horizontal strips:

$$small force = (pressure)(strip area).$$

Integrals collect those strips.

Moments and center of mass

Mass alone is not always enough. In mechanics, we often want to know how mass is distributed. For a rod on `[a, b]` with density $\varrho(x)$, the moment about the origin is

$$\int_a^b x\varrho(x)dx.$$

The center of mass is then

$$\bar{x} = (\int_a^b x\varrho(x)dx)/(\int_a^b \varrho(x)dx).$$

This pair of formulas extends the same accumulation idea: each small piece contributes not only mass, but mass weighted by location.

Example: center of mass of a variable-density rod

Let $\varrho(x) = 1 + x$ on `[0, 2]`.

The mass is

$$M = \int_0^2 (1 + x)dx = [x + x^2/2]_0^2 = 4.$$

The moment about the origin is

$$\int_0^2 x(1 + x)dx = \int_0^2 (x + x^2)dx = [x^2/2 + x^3/3]_0^2 = 2 + 8/3 = 14/3.$$

So the center of mass is

$\bar{x} = (14/3)/4 = 7/6.$

Even this new-looking formula is still just "small contribution times width, then integrate."

Hydrostatic force setup

Fluid pressure increases with depth, which means the strips near the bottom of a submerged plate contribute more force than strips near the top. A useful setup pattern is:

1. choose a horizontal strip,
2. express its depth below the fluid surface,
3. compute pressure from that depth,
4. multiply by strip area,
5. integrate over the full depth range.

This is a strong test of whether the application is being understood structurally rather than memorized mechanically.

Example: mass of a nonuniform rod

Suppose a rod on `[0, 3]` has density

$$\varrho(x) = 2 + x$$

kilograms per meter.

Then its mass is

$\int_0^3 (2 + x)dx = [2x + x^2/2]_0^3 = 6 + 9/2 = 21/2.$

So the mass is `10.5` kilograms.

Example: work against a varying force

If a spring force is $F(x) = 5x$ newtons over a stretch from $x = 0$ to $x = 2$, then

$W = \int_0^2 5x\,dx = 10$ joules.

These formulas look different on the surface, but structurally they are the same:

> Identify the small contribution, then integrate.

Quick tactics

- Sketch the region or solid before choosing the variable of integration.
- Label one representative slice and write down its width, height, radius, or area before integrating.
- Keep units visible. Area uses square units, volume uses cubic units, work uses force times distance, and mass depends on density units.
- If the geometry seems awkward, try switching from dx to dy, or from washers to shells.

Chapter review

Applications of integration are dense because they train a way of thinking, not a single formula family. The recurring blueprint is:

1. isolate a small piece,
2. express its contribution symbolically,
3. sum those contributions in the limit with an integral.

Area, volume, arc length, surface area, work, mass, center of mass, and fluid force all follow that blueprint. What changes is the expression for the small contribution.

Students who become fluent in this chapter usually stop asking "Which formula do I memorize?" and start asking "What does one thin slice contribute?" That shift is the real learning goal.

Mini projects

Project 1: slice the same region two ways

Choose a planar region that can be described using both vertical and horizontal slices. Set up the area integral both ways, explain which version is easier, and identify the algebraic cost of the less convenient choice.

Project 2: build a physical application portfolio

Create one original example each for work, mass, center of mass, and fluid force. For every example, include a sketch, unit analysis, the strip description, the integral setup, and a short note explaining why the integrand has the form it does.

Common traps

- Subtracting curves in the wrong order when finding area.
- Using a volume formula without checking whether slices or shells match the geometry.
- Forgetting the derivative inside the arc-length square root.
- Dropping units in work, mass, or density problems.
- Treating every application as a formula hunt instead of a slicing argument.

Proof window: why slicing is so general

A slice method works whenever a complicated total can be approximated by many small contributions of the form

`(simple local quantity)(small width)`.

The integral then turns a sum of tiny local pieces into an exact total. This is why the same logic appears in area, volume, work, mass, pressure, and many later topics.

Exercises

Warm-up: geometric meaning of application formulas

1. What is the general formula for area between two curves?
2. What does $A(x)$ represent in the slicing formula for volume?
3. What kind of quantity does $\varrho(x)$ represent in a mass problem?

Core skill: area, volume, mass, and work

1. Find the area between $y = x$ and $y = x^2$ on `[0, 1]`.
2. Find the volume formed by rotating $y = x$ on `[0, 1]` about the `x`-axis.
3. Find the mass of a rod on `[0, 2]` with density $\varrho(x) = 3 + x$.
4. Compute the work done by force $F(x) = 4x$ from $x = 0$ to $x = 3$.

Interpretation: shell, slice, and density reasoning

1. Explain in words why the shell method uses `radius`, `height`, and `thickness`.
2. Explain why area, work, and mass problems all have the same integral skeleton.

Challenge: setup choice and model validity

1. Describe a situation where shells are easier than washers.

2. Explain why arc length requires a square root while area between curves does not.
3. A density function is negative on part of an interval. Why would that usually signal a modeling problem?

Modeling: volume and density integrals

1. A tank is `5` meters long, and the cross-sectional area of the water at position `x` is $A(x) = 6 - x$ square meters. Write and evaluate the integral for the total water volume.
2. A cable of length `10` meters has linear density $\rho(x) = 1 + 0.2x$ kilograms per meter. Write and evaluate the integral for the cable's mass.

Reflection

Integration becomes powerful when you stop seeing formulas as isolated topics. Area, volume, work, and mass are all versions of the same idea: many small pieces, added carefully.

Chapter 10. Sequences and Series

Opening question

If you add

$1 + 1/2 + 1/4 + 1/8 + \ldots,$

does the total keep growing forever, or does it approach a finite limit?

This chapter studies infinite processes that still produce meaningful finite results. Calculus needs them because approximation, error control, and power series all depend on understanding convergence.

Learning goals

By the end of this chapter, you should be able to:

- interpret the convergence of a sequence,
- distinguish a sequence from the series built from it,
- use geometric and comparison ideas for simple series,
- understand a power series as an infinite polynomial,
- and explain the role of Taylor approximations.

Preview questions

- Why is it not enough for the terms of a series to approach zero?
- How do partial sums turn an infinite expression into something precise?
- Why can a polynomial approximate a nonpolynomial function near one point but fail far away?

10.1 Sequences and convergence

A sequence is an ordered list of numbers, usually written

$$a_1, a_2, a_3, \ldots$$

or

a_n.

The central question is whether the terms settle toward a single value as `n` grows.

If they do, the sequence converges.

Example: a convergent sequence

Consider

$a_n = 1/n$.

As `n` grows, the terms become smaller and smaller:

- `1`
- $1/2$
- $1/3$
- $1/10$
- $1/100$

The sequence converges to `0`.

Example: a divergent sequence

Let

$b_n = (-1)^n$.

Then the terms alternate:

- `-1`
- `1`
- `-1`
- `1`

The sequence does not approach a single number, so it diverges.

Why convergence matters

Convergence is the infinite-process version of stability. If the outputs settle, then later reasoning about infinite sums or approximations has something firm to stand on.

A sequence-thinking habit

When looking at a sequence, ask two questions separately:

1. What are the individual terms doing?
2. What mechanism seems to drive that behavior?

For `1/n`, the terms shrink because the denominator grows without bound. For $(-1)^n$, the terms do not settle because the sign keeps flipping. This habit matters later because the terms of a sequence and the partial sums of a series behave differently and should not be mentally merged.

10.2 Infinite series

A series is the sum of the terms of a sequence:

$suma_n.$

But an infinite series is not a completed endless addition. It is defined through its partial sums:

$S_N = a_1 + a_2 + ... + a_N.$

If the sequence of partial sums S_N converges, then the series converges.

Example: geometric series

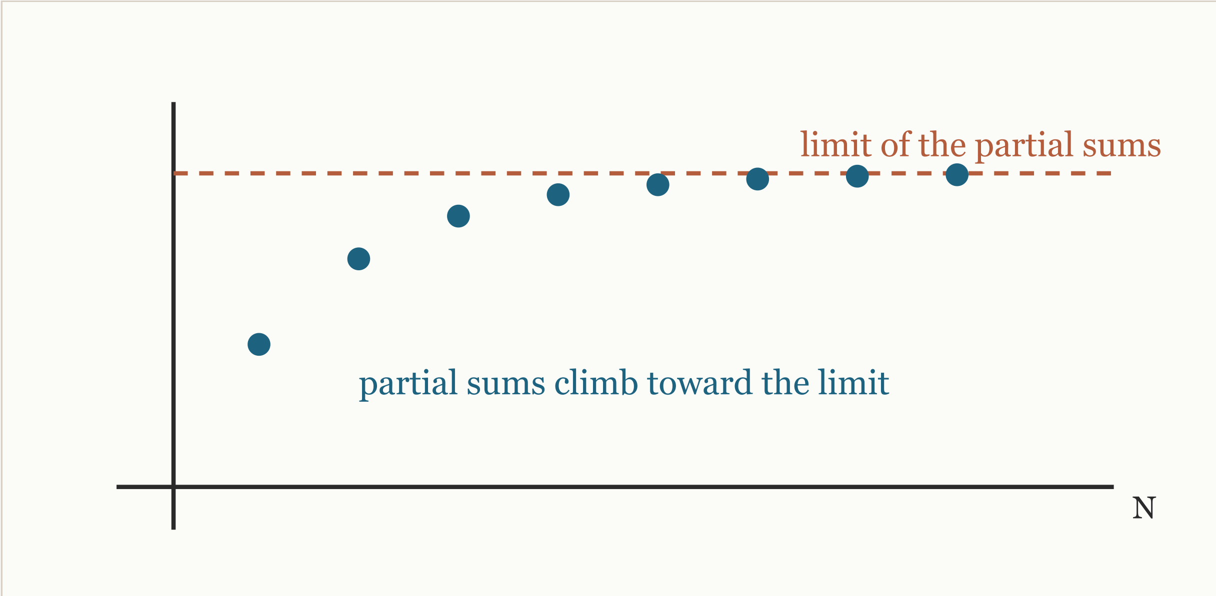
Consider

$$1 + 1/2 + 1/4 + 1/8 + ...$$

The partial sums are:

- `1`
- $3/2$
- $7/4$
- $15/8$

They keep increasing, but toward `2`.



So the infinite series converges to `2`.

Divergent series

Not every infinite sum converges.

The harmonic series

$$1 + 1/2 + 1/3 + 1/4 + ...$$

grows without bound, though very slowly. Its terms go to zero, but that alone is not enough to make the series converge.

That is a critical warning:

> For a series to converge, its terms must go to zero, but terms going to zero does not guarantee convergence.

Why partial sums matter so much

An infinite series is not interpreted by staring at the symbol

$suma_n$.

It is interpreted through the sequence

$$S_1, S_2, S_3, ...$$

of partial sums. That means every question about a series is secretly a question about a new sequence. This point is easy to overlook, but it explains why the study of sequences must come first.

10.3 Geometric series and comparison ideas

A geometric series has the form

$$a + ar + ar^2 + ar^3 + ...$$

with common ratio `r`.

If $|r| < 1$, the series converges to

`a / (1 - r)`.

If $|r| \geq 1$, it diverges.

Example

$$3 + 3/5 + 3/25 + 3/125 + ...$$

has

- $a = 3$
- $r = 1/5$

So the sum is

$3/(1 - 1/5) = 3/(4/5) = 15/4$.

Comparison idea

Sometimes a difficult series can be compared to one we already understand.

If all terms are positive and

$$0 \leq a_n \leq b_n$$

for all large `n`, then:

- if $\mathrm{sum}\, b_n$ converges, so does $\mathrm{sum}\, a_n$;
- if $\mathrm{sum}\, a_n$ diverges and $a_n \leq b_n$, then $\mathrm{sum}\, b_n$ also diverges.

This is not just a technical test. It is a strategy of bounding unknown behavior by known behavior.

A comparison mindset

Comparison tests are less about memorizing inequalities and more about pattern recognition. If a series behaves like a geometric series or a `p`-series in its tail, then comparison becomes plausible. The skill is to notice dominant behavior rather than every surface detail.

For example, a series with terms

$$(3n + 1)/(n^3 + 2)$$

behaves for large `n` like $3/n^2$, so comparison with a convergent $1/n^2$ series is natural. The exact coefficients matter less than the eventual rate of decay.

10.4 Power series

A power series is an infinite polynomial:

$\mathrm{sum} c_n (x - a)^n$.

Near the center `a`, a power series can represent a function or approximate one.

Example

The geometric series

$$1 + x + x^2 + x^3 + \ldots$$

is a power series centered at `0`. For $|x| < 1$, it sums to

`1 / (1 - x)`.

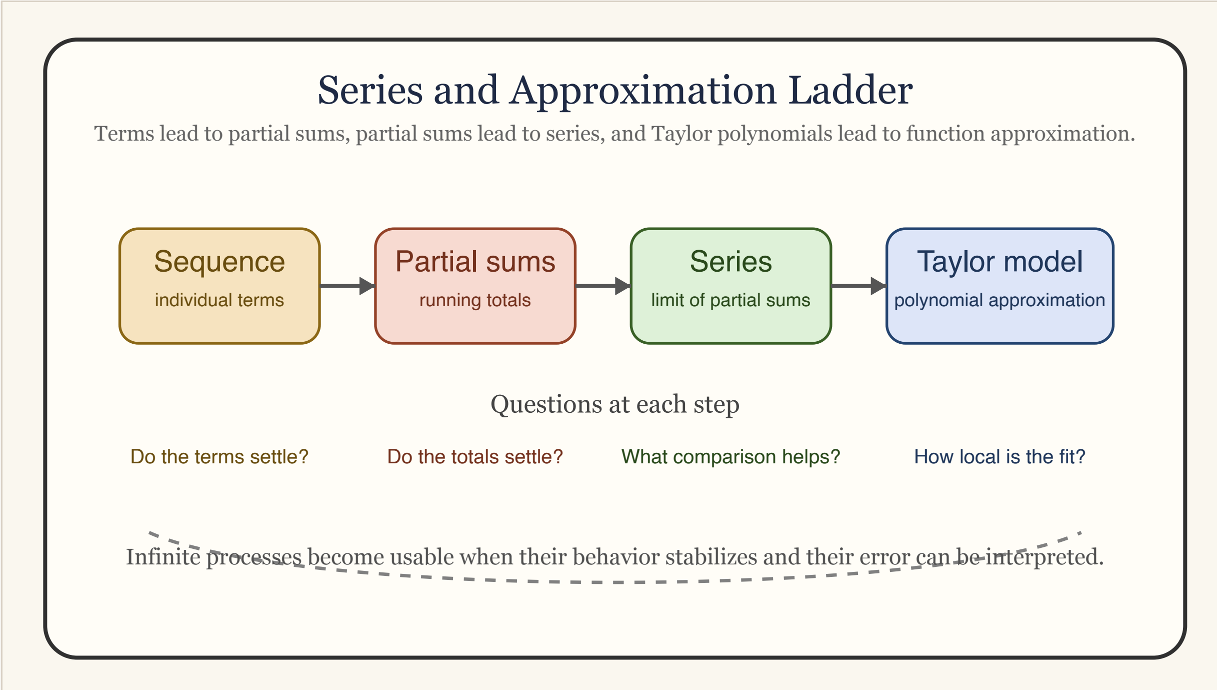
Why power series matter

Power series are flexible because polynomials are easy to differentiate, integrate, and evaluate. They let us replace a complicated function by a locally simpler infinite expression.

Radius of convergence

A power series does not usually work for every `x`. There is typically an interval around the center where it converges and outside of which it diverges.

This interval is part of the meaning of the series, not a small footnote.



Why power series are powerful

Power series matter because they behave like infinite polynomials. Inside their interval of convergence, they can often be differentiated and integrated term by term, turning hard functions into manageable algebraic objects. That is one reason so much of advanced analysis and applied mathematics is organized around series expansions.

10.5 Taylor polynomials and Taylor series

Taylor approximations use derivatives at one point to build a polynomial that locally imitates a function.

The linear approximation from earlier chapters was the first Taylor polynomial. The quadratic and cubic versions continue the same idea.

Example: quadratic approximation of e^x

Near $x = 0$,

- $e^0 = 1$
- $(e^x)' = e^x$, so the first derivative at `0` is `1`
- the second derivative at `0` is also `1`

So the quadratic Taylor polynomial at `0` is

$1 + x + x^2/2.$

For small `x`, this is a good approximation to e^x.

Why Taylor ideas belong in calculus

Taylor approximations connect derivatives, local behavior, and infinite processes. They show that a function can be replaced, near a point, by a polynomial whose coefficients encode derivative information.

This is one of the deepest recurring themes of the subject:

> complicated behavior can often be approximated locally by something simpler.

Error awareness in Taylor approximation

A Taylor polynomial is local. Saying that explicitly matters. The polynomial

$$1 + x + x^2/2$$

approximates e^x well near $x = 0$, but it becomes less reliable as `|x|` grows. A student who remembers only the formula may use it far outside its useful range. A student who remembers the local idea will ask where the approximation is centered and how far from the center the estimate is being pushed.

Worked example cluster: one function, several approximation levels

Approximate $e^{0.2}$ using:

1. the linear Taylor polynomial at `0`,
2. the quadratic Taylor polynomial at `0`,
3. the cubic Taylor polynomial at `0`.

The approximations are:

- linear: $1 + 0.2 = 1.2$
- quadratic: $1 + 0.2 + 0.2^2/2 = 1.22$
- cubic: $1 + 0.2 + 0.2^2/2 + 0.2^3/6 = 1.221333...$

The exact value is about `1.22140`, so the approximations improve as more local derivative information is included. This example shows that Taylor theory is not only abstract; it is a practical accuracy machine.

10.6 Convergence tests beyond geometric comparison

Geometric series are the first benchmark, but most interesting series are not geometric. Calculus therefore develops a family of tests that compare unknown tail behavior with known patterns.

`p`-series

One of the most important benchmarks is

$\text{sum} 1/n^p$.

This series converges when $p > 1$ and diverges when $p \leq 1$.

That means:

- $\text{sum} 1/n^2$ converges,
- $\text{sum} 1/n^3$ converges,
- $\text{sum} 1/\sqrt{n}$ diverges,
- and the harmonic series $\text{sum} 1/n$ diverges.

The `p`-series test is one of the fastest ways to classify many comparison problems.

Integral test

If $f(x)$ is positive, continuous, and decreasing for large `x`, and $a_n = f(n)$, then

$$\text{sum} a_n$$

and

$$\text{int} f(x) dx$$

have the same convergence behavior.

This test matters because it translates a series question into an integral question. For example, the integral test is a clean way to analyze the `p`-series family.

Ratio test

For a series $\text{sum} a_n$ with nonzero terms, define

$L = \lim |a_(n + 1)/a_n|$.

Then:

- if $L < 1$, the series converges absolutely,
- if $L > 1$ or the limit is infinite, the series diverges,
- if $L = 1$, the test is inconclusive.

The ratio test is especially effective when factorials or exponential powers appear.

Root test

For a series $suma_n$, define

$L = limn - throotof|a_n|$.

Then the same conclusions hold:

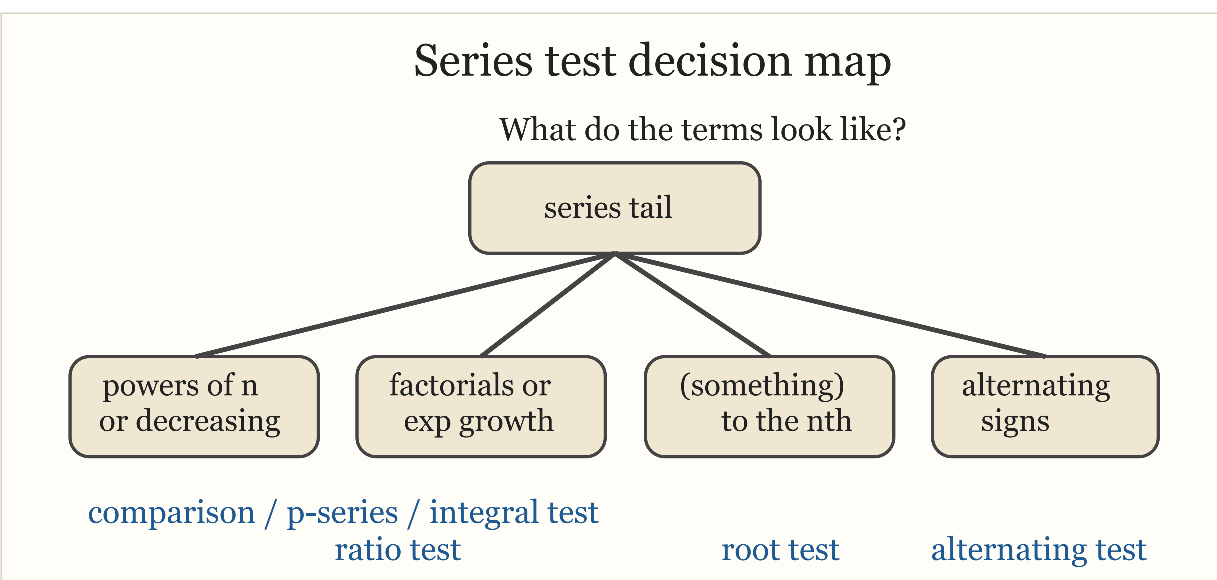
- if $L < 1$, absolute convergence,
- if $L > 1$, divergence,
- if $L = 1$, no conclusion.

The root test is often useful when each term is raised to the `n` th power.

Choosing among tests

Students often ask for a master rule that always picks the right test. No such rule exists, but some patterns are dependable:

- geometric-looking tails suggest geometric comparison,
- powers of `n` suggest `p` -series or comparison,
- factorials or exponential powers suggest ratio test,
- terms of the form $(something)^n$ suggest root test,
- positive decreasing terms built from familiar functions may suggest the integral test.



Worked example: ratio test with factorial growth

Consider

$\text{sum}\, n!/5^n.$

Then

$|a_(n+1)/a_n| = (n+1)!/5^(n+1) * 5^n/n! = (n+1)/5.$

As `n` grows, this ratio becomes arbitrarily large, so the terms do not shrink fast enough. The series diverges.

Worked example: comparison with a `p`-series

Consider

$\text{sum}(3n+1)/(n^3+2).$

For large `n`, the numerator behaves like `3n` and the denominator behaves like n^3, so the term behaves like $3/n^2$.

Because $\text{sum}\, 1/n^2$ converges, the comparison suggests convergence. That tail-thinking habit is often more important than the exact inequality details.

10.7 Alternating series, absolute convergence, and error control

Some series do not stay positive. A typical alternating series has the form

$$\text{sum}(-1)^(n-1) b_n$$

where $b_n > 0$.

Alternating Series Test

If:

- b_n decreases eventually,
- and $b_n \to 0$,

then the alternating series converges.

Example

The alternating harmonic series

$$1 - 1/2 + 1/3 - 1/4 + ...$$

converges, even though the harmonic series itself diverges.

This is a major reminder that sign patterns matter. Positive-term convergence theory and alternating convergence theory are not interchangeable.

Absolute versus conditional convergence

A series converges absolutely if

$$sum|a_n|$$

converges.

Absolute convergence is stronger than ordinary convergence. If a series converges absolutely, then it converges.

A series converges conditionally if it converges, but the absolute-value series diverges.

The alternating harmonic series is the standard example of conditional convergence.

Error estimate for alternating series

If an alternating series satisfies the test conditions, then the error made by stopping after `N` terms has magnitude at most the next omitted term:

$|error| \leq b_(N + 1).$

This is one of the most useful error estimates in an introductory calculus course because it gives accuracy information almost immediately.

Worked example: estimating an alternating sum

Approximate

$$1 - 1/2 + 1/3 - 1/4 + ...$$

using the first four terms.

The partial sum is

$1 - 1/2 + 1/3 - 1/4 = 7/12.$

The next omitted term has magnitude $1/5$, so the error is at most `0.2`. The estimate is crude, but it is guaranteed.

Why absolute convergence matters so much

Absolutely convergent series behave more robustly under rearrangement and algebraic manipulation. At this level, the most important point is conceptual: absolute convergence behaves like "safe convergence," while conditional convergence is more delicate.

10.8 Calculus with power series and Taylor remainder

Power series are especially valuable because they support calculus operations term by term inside their interval of convergence.

If

$f(x) = sum c_n (x - a)^n,$

then inside the interval of convergence we can often differentiate and integrate term by term:

- $f'(x) = sum n c_n (x - a)^(n - 1),$
- $int f(x) dx = C + sum c_n (x - a)^(n + 1)/(n + 1).$

This turns power series into a long bridge between algebra and analysis.

Standard Maclaurin series to know

At the center `0`, several benchmark series appear repeatedly:

- $e^x = 1 + x + x^2/2! + x^3/3! + ...$
- $\sin x = x - x^3/3! + x^5/5! - ...$
- $\cos x = 1 - x^2/2! + x^4/4! - ...$
- $1/(1 - x) = 1 + x + x^2 + x^3 + ...$ for $|x| < 1$

These are not just formulas to memorize. They are templates from which many other expansions can be built.

Taylor remainder as error language

A Taylor polynomial is useful only if we know something about the error. The remainder measures the gap:

$remainder = exact value - Taylor polynomial.$

At an introductory level, the most important lesson is that higher-degree polynomials usually improve local accuracy, but the improvement depends on both degree and distance from the center.

Worked example: building a local approximation for $\sin x$

Near $x = 0$,

$\sin x \approx x - x^3/6.$

At $x = 0.2$, this gives

$0.2 - 0.2^3/6 = 0.198666\ldots$

The actual value is about `0.198669...`, so the approximation is extremely accurate at that scale.

Worked example: integrating a power series

Start with

$1/(1 - x) = 1 + x + x^2 + x^3 + \ldots$ for $|x| < 1$.

Integrating term by term gives

$$-\ln(1 - x) = x + x^2/2 + x^3/3 + x^4/4 + \ldots$$

for $|x| < 1$.

This shows how one known series can generate another.

Why this chapter matters beyond calculus II

Series are not only end-of-course curiosities. They are one of the main languages of scientific approximation, numerical analysis, signal modeling, perturbation methods, and asymptotic reasoning. A dense calculus textbook should therefore treat them as central, not decorative.

Quick tactics

- Keep sequence terms and partial sums on separate mental shelves. Confusing them causes most early series errors.
- For a geometric series, inspect the common ratio first. That test is faster than expanding many terms.
- When using a Taylor polynomial, state the center point and the nearby interval where you expect the approximation to behave well.

- If a convergence test returns $L = 1$, do not force a conclusion. Switch tests.
- In alternating-series problems, check monotonic decrease and $b_n \to 0$ separately.
- When working with power series, keep the interval of convergence attached to every algebraic manipulation.

Chapter review

This chapter asks calculus to handle infinity without becoming vague. The central moves are:

- convert an infinite process into a sequence of finite approximations,
- decide whether those approximations stabilize,
- use known benchmark behaviors such as geometric decay to compare new cases,
- and use power series and Taylor polynomials to turn difficult functions into polynomial approximations.

The payoff is large. Series let calculus estimate, compute, and model functions that would otherwise be much harder to handle.

The expanded toolkit includes:

- comparison with `p`-series and integrals,
- ratio and root tests for rapid tail classification,
- alternating-series error control,
- and term-by-term calculus with power series.

Mini projects

Project 1: partial-sum laboratory

Choose three infinite series with different behaviors: one convergent geometric series, one divergent series whose terms still go to zero, and one power series evaluated at a specific input. Compute several partial sums for each, graph the running totals, and describe what the graphs reveal.

Project 2: approximation portfolio

Choose four functions and build Taylor approximations around useful center points. Compare the approximations with the exact functions at several nearby inputs and explain how accuracy changes with degree and distance from the center.

Common traps

- Confusing the terms of a sequence with the partial sums of a series.
- Assuming a series converges just because its terms go to zero.
- Forgetting the condition $|r| < 1$ for a convergent geometric series.
- Treating a power series as valid for every input.
- Thinking Taylor polynomials are exact rather than local approximations.
- Using the ratio test when $L = 1$ and pretending that means convergence.
- Forgetting that an alternating-series estimate depends on the next omitted term.
- Losing the interval of convergence after differentiating or integrating a power series.

Proof window: why the geometric sum formula works

For the finite sum

$$S_N = a + ar + ar^2 + \dots + ar^N,$$

multiplying by `r` gives

$$rS_N = ar + ar^2 + \dots + ar^{N+1}.$$

Subtract:

$$S_N - rS_N = a - ar^{N+1},$$

so

$$S_N = a(1 - r^{N+1})/(1 - r).$$

If $|r| < 1$, then r^{N+1} approaches `0` as `N` grows, leaving

`a / (1 - r)`.

Exercises

Warm-up: sequence and series language

1. What does it mean for a sequence to converge?
2. What is a partial sum?
3. When does a geometric series converge?

Core skill: convergence and geometric sums

1. Determine whether $a_n = 3/n$ converges.
2. Determine whether $b_n = (-1)^n$ converges.
3. Compute the sum of $2 + 1 + 1/2 + 1/4 + \ldots$.
4. Explain why the harmonic series is not geometric.
5. Write the first four partial sums of $1 + x + x^2 + x^3 + \ldots$ when $x = 1/2$.

Interpretation: sequence-series distinction and local approximation

1. Explain the difference between a sequence and a series in words.
2. Explain why Taylor polynomials fit the general calculus idea of local approximation.

Challenge: counterexamples and divergence logic

1. Give an example of a divergent sequence whose terms stay bounded.
2. Give an example of a series whose terms go to zero but whose sum diverges.
3. Explain why $1 + x + x^2 + \ldots$ cannot converge when $x = 2$.

Modeling: rebound totals and approximation

1. A ball loses half its rebound height on each bounce. If the first bounce reaches `3` meters, what total rebound height is accumulated over infinitely many bounces?
2. Explain why a finite Taylor polynomial can still be useful in scientific computing even when it is not exact.

Convergence tests and classification

1. Determine whether $sum\, 1/n^2$ converges.
2. Determine whether $sum\, 1/\sqrt{n}$ converges.
3. Use comparison to classify $sum(2n + 5)/(n^3 + 1)$.
4. Use comparison to classify $sum\, 1/(n^2 + 4)$.
5. Explain why the integral test is plausible for positive decreasing terms.
6. Use the ratio test to classify $sum\, n!/4^n$.
7. Use the ratio test to classify $sum\, x^n/n!$ for a fixed real `x`.
8. Explain when the root test is more natural than the ratio test.

Alternating series and absolute convergence

1. Determine whether $sum(-1)^(n - 1)/n$ converges absolutely, conditionally, or diverges.
2. Determine whether $sum(-1)^n/n^2$ converges absolutely, conditionally, or diverges.

3. State the two hypotheses of the Alternating Series Test.
4. Use the first three terms of $1 - 1/3 + 1/5 - 1/7 + \ldots$ to estimate the sum and bound the error.
5. Explain why absolute convergence is stronger than ordinary convergence.

Power series and Taylor work

1. Write the first four nonzero terms of the Maclaurin series for e^x.
2. Write the first four nonzero terms of the Maclaurin series for $\sin x$.
3. Use $1/(1 - x) = 1 + x + x^2 + \ldots$ to generate a series for $-\ln(1 - x)$.
4. Approximate $\sin(0.1)$ with $x - x^3/6$.
5. Explain why a Taylor polynomial is expected to work best near its center.
6. Describe the interval-of-convergence issue that survives after differentiating or integrating a power series.

Reflection

Sequences and series teach a powerful lesson: infinity is not chaos. Infinite processes can be disciplined, compared, and approximated. Calculus needs that discipline whenever it turns local information into global understanding.

Chapter 11. Differential Equations and Models

Opening question

If a population grows at a rate proportional to its current size, what will the population look like over time?

This chapter studies equations that describe change directly. Instead of starting with a function and differentiating it, we start with a rule for its derivative and try to recover the function itself.

Learning goals

By the end of this chapter, you should be able to:

- interpret a differential equation as a rate law,
- read a slope field,
- solve basic separable equations,
- model exponential growth and decay,
- understand the logistic model at a conceptual level,
- and explain why numerical methods are useful for differential equations.

Preview questions

- What can a slope field tell you before any formula for the solution is known?
- Why do initial conditions matter so much in differential equations?
- When should a model be trusted as descriptive, and when should it be treated as only a first approximation?

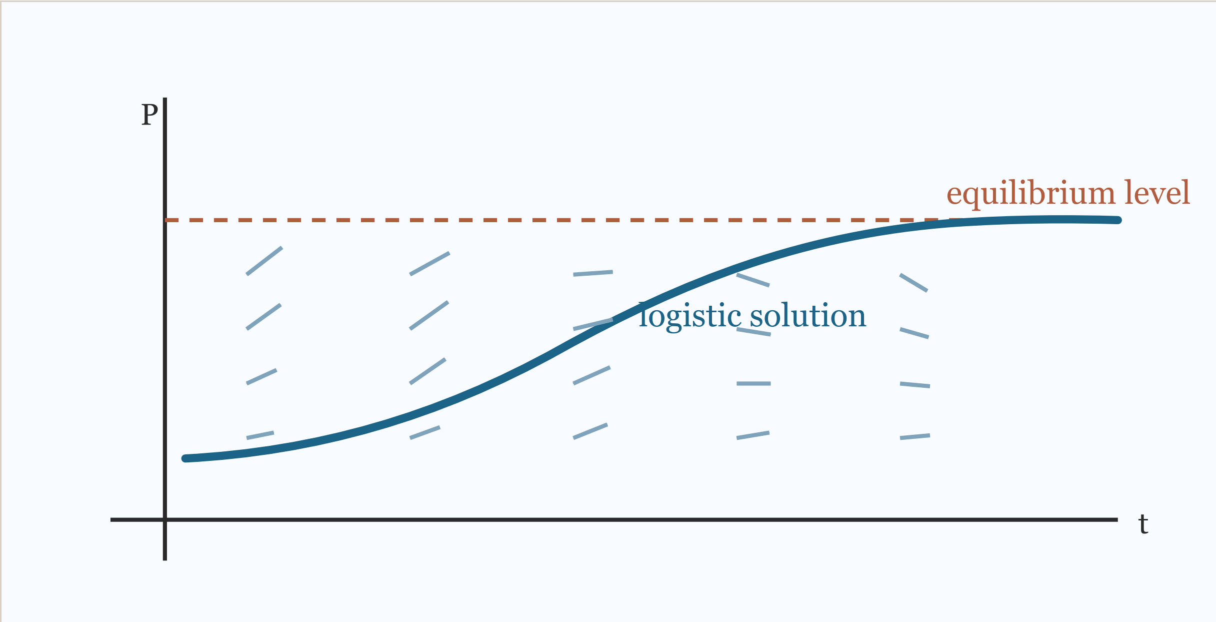
11.1 Slope fields

A differential equation gives a rule for slope. For example,

$$dy/dx = x - y$$

tells us the slope of the solution curve at each point `(x, y)`.

A slope field is a picture of those local directions. At many points in the plane, we draw a tiny line segment whose slope matches the differential equation there.



Why slope fields matter

Before solving a differential equation exactly, a slope field lets us see:

- where solutions rise or fall,
- where they flatten,
- and what long-term behavior is plausible.

This is important because exact symbolic solutions are not always available.

Reading a slope field strategically

Good slope-field reading focuses on patterns rather than isolated line segments. Ask:

- where are the slopes zero,
- where are they positive or negative,
- where are they steep,
- and which curves seem to act like long-term targets or barriers?

These questions often reveal equilibrium behavior even before any solution formula is written down.

11.2 Separable equations

A differential equation is separable if it can be rewritten so that all `y` terms appear with dy and all `x` terms appear with dx.

Example

Solve

$dy/dx = 2x.$

This is already easy to separate:

$dy = 2x dx.$

Integrate both sides:

$int dy = int 2x dx,$

so

$y = x^2 + C.$

Example with `y`

Solve

$dy/dx = xy.$

Separate:

$(1/y) dy = x dx.$

Integrate:

$int(1/y) dy = int \times dx.$

Then

$\ln|y| = x^2/2 + C.$

Exponentiating gives a family of solutions.

The exact algebra after exponentiation matters less at first than the core method:

> isolate the variables, integrate both sides, and interpret the constant.

A separable-equation checklist

When solving a separable differential equation, keep the following order:

1. rewrite so `y`-terms and `x`-terms are separated,
2. integrate both sides carefully,
3. simplify enough to apply the initial condition,
4. interpret the resulting constant in context.

Students often mix steps `2` and `3`, which can make the constant hard to track.

11.3 Exponential growth and decay

If a quantity grows at a rate proportional to itself, the model is

$dy/dt = ky.$

The solutions are exponential:

$y = Ce^{kt}.$

Growth

If $k > 0$, the quantity grows.

Examples:

- idealized population growth,
- compound interest in continuous form,
- early stages of some epidemics.

Decay

If $k < 0$, the quantity decays.

Examples:

- radioactive decay,
- cooling toward a surrounding temperature in simplified models,
- drug concentration decrease in simple compartments.

Example

A population satisfies

$$P = 0.04P$$

with $P(0) = 200$.

Then

$P(t) = 200e^{0.04t}$.

The derivative law already told us the structure of the answer before we solved it.

Doubling time and half-life

Exponential models become much more useful when the parameters are interpreted. If $y = Ce^{kt}$, then:

- a doubling time solves $e^{kt} = 2$,
- a half-life solves $e^{kt} = 1/2$.

In both cases, logarithms convert a rate constant into an observable time scale. This is one reason exponential models appear so often in science: the parameter `k` has a clear physical meaning.

11.4 Logistic models

Pure exponential growth cannot continue forever in most real systems. Resources, space, and competition introduce limits.

The logistic model modifies exponential growth:

$dP/dt = kP(1 - P/M)$,

where:

- `k` is a growth parameter,
- `M` is the carrying capacity.

What the model says

- When `P` is small relative to `M`, the factor `(1 - P/M)` is close to `1`, so the model looks almost exponential.
- When `P` is near `M`, the factor becomes small, so growth slows.
- If $P = M$, the derivative is `0`, so `M` is an equilibrium level.

Why logistic models matter

The logistic model is a good example of calculus as modeling:

- write a plausible local rate law,
- analyze the consequences,
- and compare the predictions with real behavior.

Not every logistic model is accurate, but it teaches a powerful habit: think first about how the rate should depend on the current state.

Equilibrium analysis

The logistic model has two equilibrium values:

- $P = 0$
- $P = M$

Those values are special because they make the derivative zero. Around them, the sign of the derivative shows how nearby solutions move:

- for $0 < P < M$, the derivative is positive, so solutions rise,
- for $P > M$, the derivative is negative, so solutions fall back,
- so `M` acts like a stable long-term level in the model.

This kind of sign analysis is one of the most valuable habits in differential equations because it gives global behavior without solving the equation completely.

11.5 Numerical solution ideas

Many differential equations cannot be solved neatly with elementary functions. Numerical methods estimate solution values step by step.

Euler's Method

Start at a known point. Use the derivative there to take a short linear step:

`next value approx current value + slope * step size`.

Example idea

If

$$y = x + y$$

and $y(0) = 1$, then with a small step `h`,

$y(0 + h) \approx 1 + (0 + 1)h = 1 + h$.

Then the process can be repeated.

A short Euler table

Take $h = 0.1$ for the initial value problem

$y' = x + y, y(0) = 1.$

Then:

- at `(0, 1)`, slope = `1`, so $y(0.1) \approx 1 + 0.1(1) = 1.1$
- at `(0.1, 1.1)`, slope = `1.2`, so $y(0.2) \approx 1.1 + 0.1(1.2) = 1.22$

Even this tiny table reveals the philosophy of numerical solution: a differential equation can generate a curve step by step from local slope information alone.

Euler's Method is not exact, but it matches the same local-linear philosophy used throughout the book:

- use current slope,
- move a short distance,
- update,
- repeat.

Why step size matters

A smaller step size usually improves the estimate because the tangent-line assumption has less time to drift away from the true curve. This is another reminder that local linearity is strongest on short intervals.

Higher-order numerical methods improve this basic idea by using more slope information on each step, but Euler's Method remains the cleanest first model of what numerical solution really means.

Why numerical methods belong here

Numerical methods are not a side topic. They are how differential equations are handled in much of science and engineering.

A modern calculus book should make room for that reality.

11.6 Autonomous equations and phase lines

An autonomous differential equation has the form

$dy/dt = f(y)$.

The derivative depends only on the current state `y`, not explicitly on time.

Why autonomous equations matter

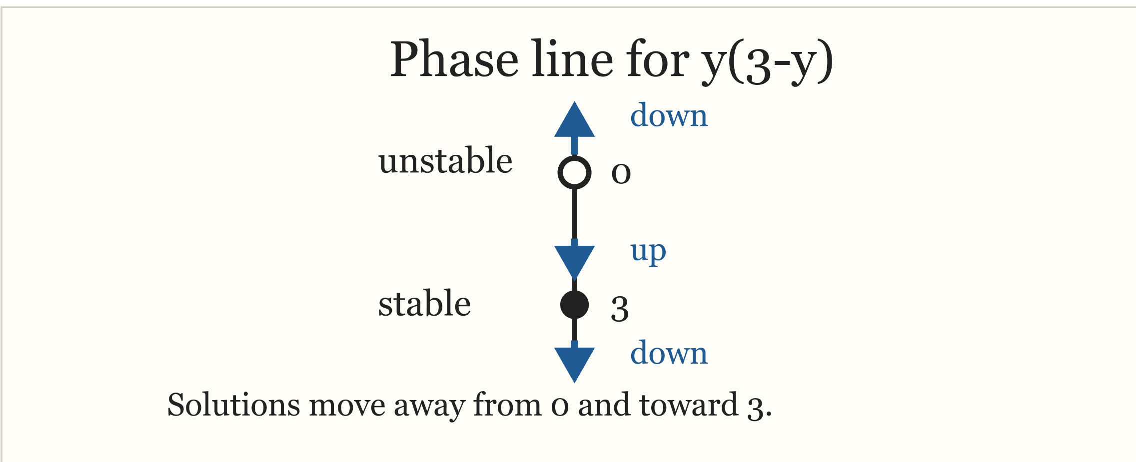
These equations are ideal for long-term qualitative analysis. Instead of chasing exact formulas immediately, we can study the sign of $f(y)$ and identify equilibrium values.

Phase line idea

If $f(y) = 0$, the state is at equilibrium. Between equilibria, the sign of $f(y)$ tells whether solutions move upward or downward.

- if $f(y) > 0$, solutions rise,
- if $f(y) < 0$, solutions fall.

This sign information is displayed on a phase line.



Worked example

Consider

$dy/dt = y(3 - y)$.

The equilibria are $y = 0$ and $y = 3$.

- for $y < 0$, `y` is negative and `3-y` is positive, so $dy/dt < 0$,
- for $0 < y < 3$, both factors are positive, so $dy/dt > 0$,
- for $y > 3$, `y` is positive and `3-y` is negative, so $dy/dt < 0$.

So solutions move away from `0` and toward `3`. The equilibrium `0` is unstable, while `3` is stable.

Why phase lines are powerful

A phase line compresses a great deal of global information into a one-dimensional picture:

- equilibrium values,
- stability,
- long-term behavior,
- and the direction of solution motion.

This is one of the clearest examples of calculus extracting global understanding from local derivative data.

11.7 Cooling, mixing, and one-compartment models

First-order differential equations appear in many applied settings beyond population growth.

Newton's Law of Cooling

If $T(t)$ is the temperature of an object in an environment of temperature T_s, then a common model is

$$dT/dt = -k(T - T_s).$$

The rate of change depends on the difference between the object's temperature and the surrounding temperature.

What the sign means

- if the object is warmer than the surroundings, then $T - T_s > 0$ and the derivative is negative, so the object cools,
- if the object is cooler than the surroundings, the derivative is positive, so the object warms.

The model drives temperature toward the ambient value T_s.

Worked example

A cup of coffee at `90` degrees Celsius is placed in a room at `20` degrees Celsius. If the model is

$$T' = -0.15(T - 20),$$

then the equilibrium temperature is `20`. The farther the coffee is above room temperature, the faster it cools.

Mixing and drug models

The same structural idea appears in:

- mixing problems, where concentration changes depend on inflow and outflow,
- pharmacokinetic models, where drug concentration decays proportionally to current amount,
- and charge-discharge problems in simple circuits.

These are not identical models, but they share a central feature: the derivative law reflects a balance of present-state effects.

Why this matters in a calculus course

A strong calculus book should show students that the symbolic techniques are not isolated tricks. The same first-order logic reappears across physics, chemistry, biology, and engineering.

11.8 Improved Euler, model accuracy, and trust

Euler's Method uses one slope per step. A natural refinement is to sample more than one slope before updating.

Improved Euler idea

One common version is:

1. use the current slope to make a short prediction,
2. evaluate the slope again at the predicted point,
3. average the two slopes,
4. update with that average.

Even without mastering the full formula, students should see the idea: better numerical methods reduce error by sampling more of the local behavior.

Local and global error

A single Euler step makes a local error because the true curve bends away from the tangent approximation. Repeating many steps accumulates error across the interval.

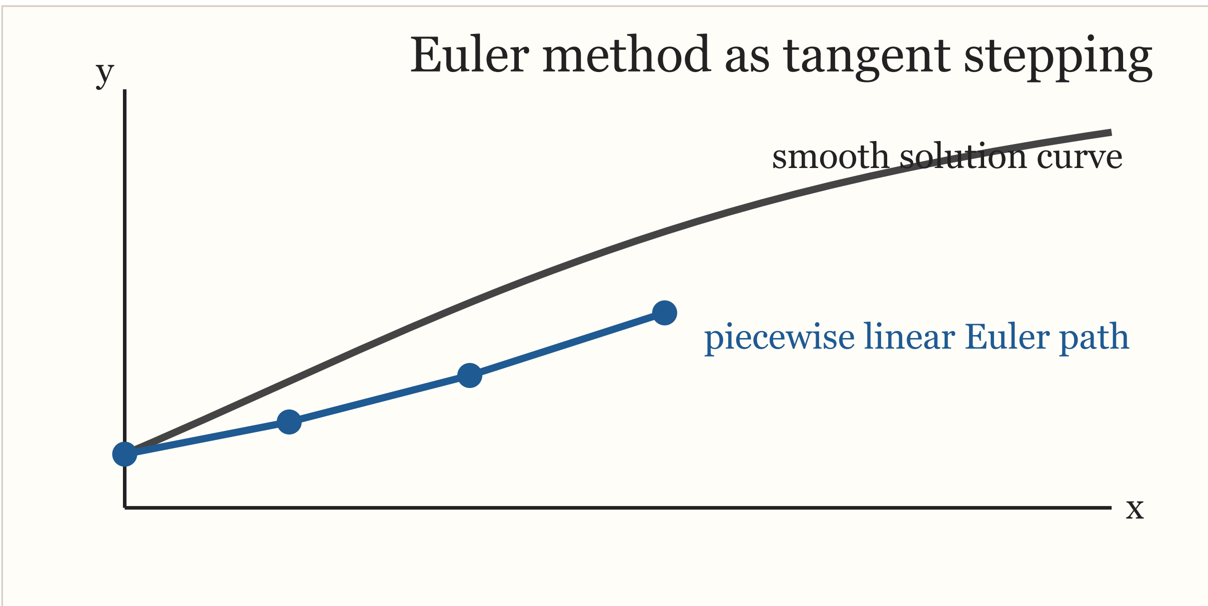
Worked example: one Euler step versus a refined estimate

For

$y' = x + y, y(0) = 1$, with $h = 0.2$:

- Euler uses slope `1` at `(0,1)` and predicts $y(0.2) \approx 1.2$,
- a refined method notices that the slope near the endpoint is larger than `1`, so a better estimate should be a bit above `1.2`.

That reasoning matters even before formulas do. Good numerical work includes qualitative sanity checks.



Trusting a model

A model can fail for at least three reasons:

- the differential equation is a poor description of the system,
- the parameter values are poorly estimated,
- or the numerical method is too crude for the chosen step size.

This is why modern quantitative work combines symbolic reasoning, numerical approximation, parameter estimation, and critical interpretation.

Why differential equations are central

Differential equations are one of the clearest places where calculus stops being only a collection of methods and becomes a language for systems, feedback, and prediction.

Quick tactics

- Read every differential equation as a statement about rate before treating it as an algebra problem.
- In a model, identify equilibrium values early; they often explain the long-term story faster than explicit formulas do.
- Use slope fields for qualitative information and separable methods for exact formulas when available.
- In Euler's Method, keep the step size and the update rule visible in a table.
- For autonomous equations, do a sign analysis before looking for a formula.
- In cooling and population models, identify the equilibrium quantity in words.
- When numerical results look unreasonable, check the step size before trusting the arithmetic.

Chapter review

Differential equations shift the focus of calculus from functions themselves to laws of change. That shift is powerful because many scientific descriptions are naturally local:

- growth depends on current population,
- cooling depends on temperature difference,
- chemical concentration depends on present concentration,
- and feedback-limited growth depends on both current size and capacity.

This chapter therefore belongs near the center of a calculus text. It shows how derivative ideas become models.

The expanded toolkit now includes:

- phase-line analysis for autonomous equations,
- temperature, mixing, and one-compartment interpretations,
- and a clearer distinction between model error and numerical error.

Mini projects

Project 1: slope-field storytelling

Choose a first-order differential equation, generate or sketch a slope field, and write a short qualitative report describing likely solution behavior before solving the equation exactly. Then compare the prediction with the exact or numerical solution.

Project 2: model comparison

Choose one scenario, such as population growth or temperature change, and compare an exponential model with a logistic or offset model. Identify what each model captures well and what each one misses.

Common traps

- Treating a differential equation as only an algebraic puzzle rather than a rate law.
- Separating variables incorrectly.
- Forgetting the initial condition when solving for the constant.
- Assuming exponential growth is a good model forever.
- Treating a numerical approximation like an exact symbolic solution.
- Forgetting that an equilibrium must make the derivative zero.
- Confusing the ambient temperature with the object's current temperature in a cooling model.
- Believing a numerical table without checking whether its trend matches the qualitative slope information.

Proof window: why $dy/dt = ky$ leads to exponentials

If the rate of change of a quantity is always proportional to the quantity itself, then the function must keep reproducing its own shape under differentiation. Exponential functions do exactly that.

This is why they appear so naturally in growth and decay models. The equation itself points to the family of solutions.

Exercises

Warm-up: slope fields and separability

1. What does a slope field represent?
2. What does it mean for a differential equation to be separable?
3. In the logistic model, what is the carrying capacity?

Core skill: basic differential-equation solving

1. Solve $dy/dx = 3x$.

2. Solve $dy/dx = 4y$.
3. Find the solution of $P' = 0.1P$ with $P(0) = 50$.
4. Explain whether the solution of $y' = -2y$ grows or decays.

Interpretation: exponential and logistic behavior

1. Explain the difference between an exponential model and a logistic model.
2. Explain why Euler's Method is related to tangent lines and local linearity.

Challenge: equilibrium and qualitative analysis

1. Give an example of a context where logistic growth is more realistic than exponential growth.
2. Explain why a slope field can be useful even when no exact solution formula is known.
3. A differential equation has zero slope whenever $y = 3$. What does that suggest about the line $y = 3$?

Modeling: cooling and population

1. A cooled object satisfies $T' = -0.2(T - 20)$. Explain what the number `20` means in the model.
2. A population follows $P' = 0.5P(1 - P/1000)$. Explain in words what happens when $P = 100$, $P = 900$, and $P = 1000$.

Autonomous equations and phase lines

1. Find the equilibria of $dy/dt = y(4 - y)$.
2. Determine the sign of dy/dt for the intervals $y < 0$, $0 < y < 4$, and $y > 4$ in problem 15.
3. Identify which equilibria in problem 15 are stable and which are unstable.
4. Explain why a phase line can predict long-term behavior without solving the equation explicitly.

Cooling, decay, and modeling structure

1. Write a differential equation for Newton cooling toward room temperature `18`.
2. Explain why the ambient temperature is an equilibrium in Newton's law of cooling.
3. A drug amount $A(t)$ satisfies $A' = -0.3A$. Interpret the meaning of `0.3`.
4. Explain why exponential decay is a reasonable first model for one-compartment drug elimination.
5. Describe a situation where a simple cooling or decay model would be too crude.

Numerical methods and trust

1. Use one Euler step with $h = 0.1$ for $y' = x + y$, $y(0) = 1$, to estimate $y(0.1)$.
2. Use a second Euler step from problem 24 to estimate $y(0.2)$.
3. Explain why smaller step sizes usually improve Euler approximations.
4. Explain the difference between model error and numerical error.
5. Describe a sign or trend check you could use to judge whether a numerical solution table is plausible.

Reflection

Differential equations put calculus into its most natural modeling form. Instead of asking only "what is the function?" they ask "how does the function change?" From that local law, global behavior can often be reconstructed.

Chapter 12. Vectors and Space

Opening question

How do we describe motion when one number is not enough?

A point moving through space needs more than a position on a line. It needs coordinates, direction, and often a geometric language for length, angle, and area. This chapter extends calculus into space by building that language.

Learning goals

By the end of this chapter, you should be able to:

- represent points and vectors in three-dimensional space,
- describe curves parametrically,
- interpret velocity and acceleration in space,
- use dot products for angle and projection,
- and use cross products to measure oriented area.

Preview questions

- What additional geometry appears when a curve is allowed to move through space instead of a plane?
- Why do lines in space need direction vectors while planes need normal vectors?
- How do dot and cross products answer different geometric questions?

12.1 Points, vectors, and geometry in space

A point in space is described by coordinates `(x, y, z)`. A vector records displacement or direction and is often written

$\langle a, b, c \rangle$.

Length

The length of $\langle a, b, c \rangle$ is

$\sqrt{a^2 + b^2 + c^2}$.

This is the three-dimensional version of the Pythagorean Theorem.

Vector arithmetic

Vectors can be:

- added,
- scaled by constants,
- interpreted as displacements from one point to another.

If $P = (1, 2, 0)$ and $Q = (4, 3, 5)$, then the displacement from `P` to `Q` is

$\langle 3, 1, 5 \rangle$.

Vectors turn geometry into a language calculus can use.

Unit vectors and direction

A unit vector has length `1`. Unit vectors are valuable because they separate direction from magnitude. If `v` is nonzero, then

`v / |v|`

points in the same direction as `v` but has length `1`.

This becomes essential in motion, projection, and later in normal-vector geometry.

Distance between points in space

The distance from $P = (x_1, y_1, z_1)$ to $Q = (x_2, y_2, z_2)$ is the length of the displacement vector:

$sqrt((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2)$.

This is the three-dimensional distance formula, and it is often the first sign that familiar plane geometry survives in space with only modest changes.

12.2 Lines, planes, and curves in space

The simplest geometric objects in space are lines and planes.

Lines in space

A line through a point r_0 with direction vector `v` can be written in vector form as

$$r(t) = r_0 + tv.$$

This says: start at the base point and move any scalar multiple of the direction vector.

Example: line through a point

Suppose a line passes through `(1, 2, 3)` in direction $\langle 2, -1, 4 \rangle$. Then

$$r(t) = \langle 1, 2, 3 \rangle + t\langle 2, -1, 4 \rangle.$$

In parametric form, this becomes

- $x = 1 + 2t$
- $y = 2 - t$
- $z = 3 + 4t$

Planes in space

A plane is naturally described by a point together with a normal vector. If $n = \langle a, b, c \rangle$ is normal to the plane and the plane passes through (x_0, y_0, z_0), then

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

The normal vector is perpendicular to every direction lying in the plane.

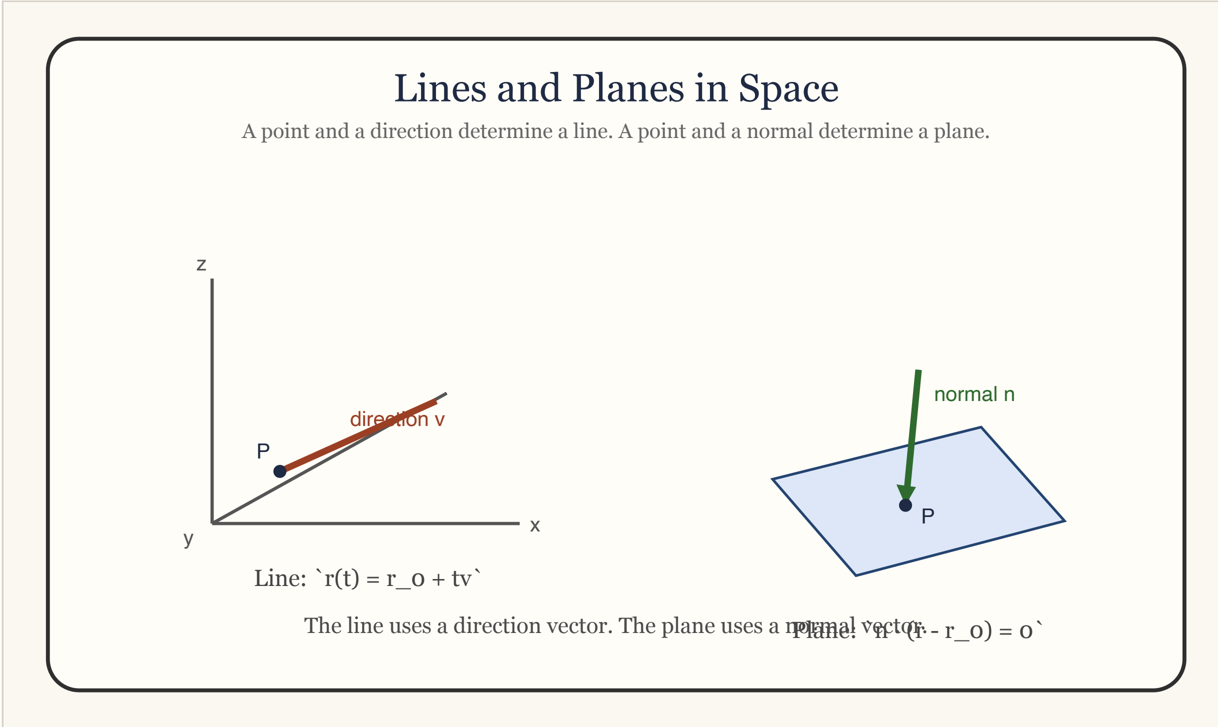
Example: plane through a point

If a plane passes through `(1, -1, 2)` and has normal vector $\langle 2, 3, -1 \rangle$, then its equation is

$$2(x - 1) + 3(y + 1) - (z - 2) = 0.$$

Simplifying gives

$$2x + 3y - z + 3 = 0.$$



A curve in space is often described parametrically:

$r(t) = \langle x(t), y(t), z(t) \rangle$.

The parameter `t` is frequently time, but it can be any running variable.

Example: a helix

$$r(t) = \langle \cos t, \sin t, t \rangle$$

traces a spiral rising around the `z`-axis.

This kind of description is powerful because it separates the motion of each coordinate while still producing one object in space.

Why parametrization matters

Parametric descriptions are natural when:

- an object is moving,
- a path is built from a process,
- or the curve fails to be the graph of a single function $z = f(x, y)$.

Space curves versus surfaces

It is useful to distinguish:

- a curve, which is traced by one parameter,

- and a surface, which usually requires two parameters or an implicit equation.

That distinction becomes important later when line and surface integrals appear in vector calculus.

12.3 Motion along a curve

If

$r(t) = \langle x(t), y(t), z(t) \rangle$,

then

- `r'(t)` is velocity,
- `r''(t)` is acceleration.

Example

Let

$r(t) = \langle t, t^2, t^3 \rangle$.

Then

- $r'(t) = \langle 1, 2t, 3t^2 \rangle$
- $r''(t) = \langle 0, 2, 6t \rangle$

The velocity vector tells both the direction and the speed of motion. Its magnitude is the speed:

`|r'(t)|`.

Tangent vectors

The derivative vector is tangent to the path. This is the space-curve version of the tangent line idea from single-variable calculus.

Speed from velocity

The velocity vector contains direction, but the speed is a scalar:

$speed = |r'(t)|$.

This distinction matters in modeling. A particle may have large speed while changing direction rapidly, or small speed while moving along a straight segment.

Arc length along a parametric curve is built from that speed, which shows that the one-variable ideas of accumulation and local approximation still apply in spatial motion.

Example: speed along a helix

For

$r(t) = \langle \cos t, \sin t, t \rangle,$

the velocity is

$r'(t) = \langle -\sin t, \cos t, 1 \rangle.$

Its speed is

$|r'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}.$

So this motion has constant speed even though its direction changes continuously. That is a useful reminder that speed and direction are genuinely separate pieces of information.

Tangent lines to space curves

At time $t = a$, the tangent line to a space curve uses the point $r(a)$ and the tangent vector `r'(a)`:

$L(s) = r(a) + s r'(a).$

This construction parallels the tangent line idea from single-variable calculus, but the resulting line now lives in three-dimensional space.

12.4 Dot product and projection

The dot product of vectors $u = \langle a, b, c \rangle$ and $v = \langle d, e, f \rangle$ is

$u \cdot v = ad + be + cf.$

It is useful because it links coordinates to angle:

$u \cdot v = |u||v| \cos \theta.$

Orthogonality

If $u \cdot v = 0$, then the vectors are perpendicular.

Projection

The component of `u` in the direction of `v` is built from the dot product. This matters whenever we want:

- work done by a force along a direction,
- shadows or projections,
- or decomposition into parallel and perpendicular parts.

Example

If

$$u = \langle 2, 1, 0 \rangle$$

and

$v = \langle 1, 0, 0 \rangle$,

then

$u \cdot v = 2$,

so the `x`-direction component of `u` is `2`.

Angle interpretation

Because the dot product contains $\cos\theta$, it can be used to recover the angle between two nonzero vectors. This is useful in mechanics, graphics, and geometry whenever alignment matters.

Projection as decomposition

One reason the dot product matters so much is that it lets a vector be split into:

- a component parallel to a chosen direction,
- and a component perpendicular to that direction.

That decomposition is the geometric heart of work, shadows, least-squares ideas, and orthogonality methods that appear later in mathematics.

12.5 Cross product and area

The cross product of two vectors in space is another vector:

`u x v`.

Its magnitude is

$|u \times v| = |u||v| \sin \theta.$

This makes it a natural area tool.

Geometric meaning

The magnitude `|u x v|` equals the area of the parallelogram spanned by `u` and `v`.

Half of that gives the area of the triangle they form.

Orientation

Unlike the dot product, the cross product depends on order. Reversing the order changes the direction:

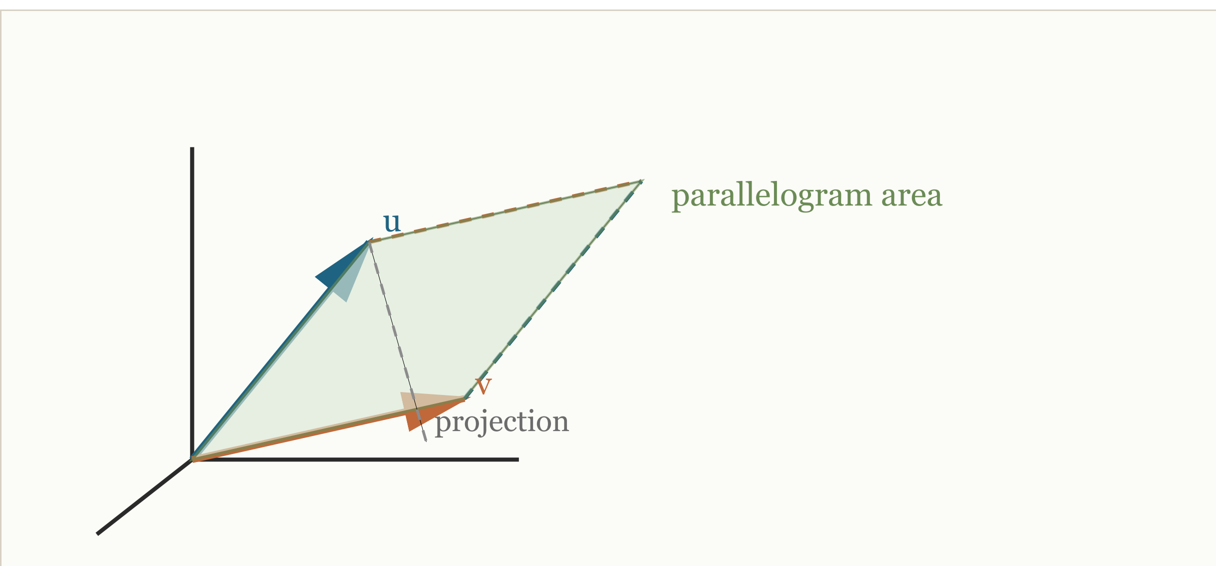
$v \times u = -(u \times v).$

This orientation behavior becomes important later in flux and Stokes' Theorem.

A concrete area example

If $u = \langle 1, 0, 0 \rangle$ and $v = \langle 0, 2, 0 \rangle$, then `u x v` has magnitude `2`. That matches the area of the rectangle spanned by those perpendicular vectors.

When the vectors are not perpendicular, the cross product still measures the slanted parallelogram area directly, which is why it is so useful in geometry and physics.



Cross products and normals

If `u` and `v` lie in a plane, then `u x v` is perpendicular to that plane. This makes the cross product a natural way to build normal vectors from geometric data.

That fact becomes especially important later in surface integrals, where a normal direction must be chosen for a parameterized surface.

Quick tactics

- Keep the distinction between point and vector explicit; a point marks location, a vector records displacement or direction.
- Use a unit vector when the problem asks for direction only.
- For lines, think "point plus direction." For planes, think "point plus normal."
- Treat the dot product as the angle/projection tool and the cross product as the area/orientation tool.

Chapter review

This chapter supplies the geometric grammar for multivariable and vector calculus. The main pieces are:

- vectors as directed quantities in space,
- parametric curves as one-parameter motions,
- velocity and acceleration as derivatives of those motions,
- dot products as tools for angle and projection,
- and cross products as tools for area and orientation.

Without this chapter, the later geometry of flux, circulation, and surfaces would have no language to stand on.

Mini projects

Project 1: spatial motion profile

Choose or invent a parametric space curve for a drone, roller coaster, or satellite path. Compute velocity, acceleration, and speed, then explain what the motion looks like geometrically.

Project 2: line-plane geometry notebook

Build five small examples of lines and planes in space. For each one, identify the relevant point, direction or normal vector, and a short explanation of what geometric information that vector carries.

Common traps

- Mixing up points and vectors.
- Forgetting that velocity in space is a vector, not just a speed.
- Treating the dot product like a cross product or vice versa.
- Forgetting that `u x v` changes sign when the order is reversed.
- Using a parameterization without interpreting what the parameter means.

Proof window: why the dot product detects angle

For vectors based at the origin, the Law of Cosines gives a relation among side lengths of the triangle they form. Expanding that relation in coordinates leads to

$u \cdot v = |u||v| \cos \theta.$

So the dot product is not merely a coordinate recipe. It is a compact encoding of angle information.

Exercises

Warm-up: vectors and motion

1. What does a vector represent geometrically?
2. What does `r'(t)` represent for a space curve?
3. What does it mean if $u \cdot v = 0$?

Core skill: vector computation and curve derivatives

1. Find the length of $\langle 3, 4, 12 \rangle$.
2. Find the displacement from `(1, 0, 2)` to `(5, 3, 1)`.
3. Differentiate $r(t) = \langle t^2, \sin t, e^t \rangle$.
4. Compute $\langle 1, 2, 3 \rangle \cdot \langle 4, 0, -1 \rangle$.

Interpretation: speed, velocity, and geometry

1. Explain the difference between speed and velocity for a space curve.
2. Explain why the cross product is naturally tied to area.

Challenge: orthogonality and path reasoning

1. Give an example of two nonzero perpendicular vectors in space.
2. Explain why a parametric curve can represent a path that is not the graph of a function $z = f(x, y)$.
3. A particle has $r'(t) = \langle 0, 0, 0 \rangle$ at one instant. What does that mean physically?

Modeling: motion and work

1. A drone's position is $r(t) = \langle 2t, 3t, 10 - t^2 \rangle$. Find its velocity and acceleration.
2. A force vector and a displacement vector are perpendicular. What does that imply about the work component in that direction?

Reflection

Vectors give calculus a spatial grammar. Instead of tracking one changing number, we can now track positions, directions, and geometric quantities in space.

Chapter 13. Multivariable Functions

Opening question

A weather map assigns a temperature to each point on a city map. A mountain assigns an elevation to each point on the ground below it.

How do we study change when the output depends on more than one input at once?

This chapter generalizes the central ideas of calculus to functions of several variables.

Learning goals

By the end of this chapter, you should be able to:

- interpret surfaces and contour maps,
- understand limits and continuity in several variables at an intuitive level,
- compute partial derivatives,
- interpret the gradient,
- and build tangent-plane approximations.

Preview questions

- Why can multivariable limits fail even when many tested paths appear to agree?
- What does it mean to hold one variable fixed while another changes?
- How does the gradient combine many possible directional changes into one vector?

13.1 Surfaces and contour maps

A function of two variables has the form

$z = f(x, y).$

Its graph is a surface in three-dimensional space.

Example

$$f(x,y) = x^2 + y^2$$

produces an upward-opening bowl.

Contour maps

Instead of graphing the whole surface, we can look at level curves:

$f(x,y) = c.$

These are contour lines. On a topographic map, each contour line represents a constant height.

Contour maps are useful because they compress three-dimensional information into a two-dimensional picture.

Reading contour spacing

Contour maps carry more than altitude labels. Closely spaced contours suggest rapid change, while widely spaced contours suggest gentler change. This is the multivariable analogue of a graph becoming steeper in one-variable calculus.

13.2 Limits and continuity in several variables

In one-variable calculus, a limit asks what happens as `x` approaches `a` along a line.

In several variables, there are many possible approach paths.

That makes multivariable limits more subtle.

Example idea

Consider

$$f(x,y) = (xy)/(x^2 + y^2)$$

near `(0, 0)`.

If you approach along $y = x$, the expression becomes

$x^2/(2x^2) = 1/2.$

If you approach along $y = -x$, it becomes

$-x^2/(2x^2) = -1/2.$

Because different paths give different values, the limit does not exist.

Continuity

A multivariable function is continuous at a point when nearby inputs produce nearby outputs in a stable way, independent of path.

Polynomials in several variables are continuous everywhere, just as in one variable.

Why path tests are only a first tool

If two different paths give different values, the limit definitely fails. But if several tested paths give the same value, that still does not prove the limit exists. Multivariable limits often require more than a few sample paths.

A path-testing habit

When searching for path dependence, try:

- straight lines such as $y = mx$,
- coordinate axes,
- curves such as $y = x^2$,
- and any path suggested by the algebra of the expression.

These tests are fast ways to expose inconsistency, though never complete evidence of existence.

13.3 Partial derivatives

A partial derivative measures change with respect to one input while the others are held fixed.

For $f(x, y)$, the first partial derivatives are:

- f_x
- f_y

Example

If

$f(x, y) = x^2y + 3y^2,$

then

- $f_x = 2xy$
- $f_y = x^2 + 6y$

Interpretation

- f_x tells how the function changes when `x` changes but `y` is frozen.
- f_y tells how the function changes when `y` changes but `x` is frozen.

This is not the whole story of change, but it is the beginning.

Mixed influence

In multivariable settings, changing one variable may matter differently depending on the value of the other. That is why partial derivatives are themselves functions of several variables.

A physical reading of partial derivatives

If $T(x, y)$ is temperature on a metal plate, then:

- T_x measures how temperature changes when moving east-west while holding north-south position fixed,
- T_y measures how temperature changes when moving north-south while holding east-west position fixed.

This language helps prevent a common mistake: students sometimes treat partial derivatives as formal symbols without noticing that each one encodes a different experiment.

13.4 The gradient

The gradient of `f` is the vector

$\nabla f = \langle f_x, f_y \rangle.$

It points in the direction of steepest increase.

Why this matters

The gradient combines the separate partial-derivative directions into one geometric object. It tells us:

- which way the function rises fastest,

- and how large that fastest rate is.

Example

If

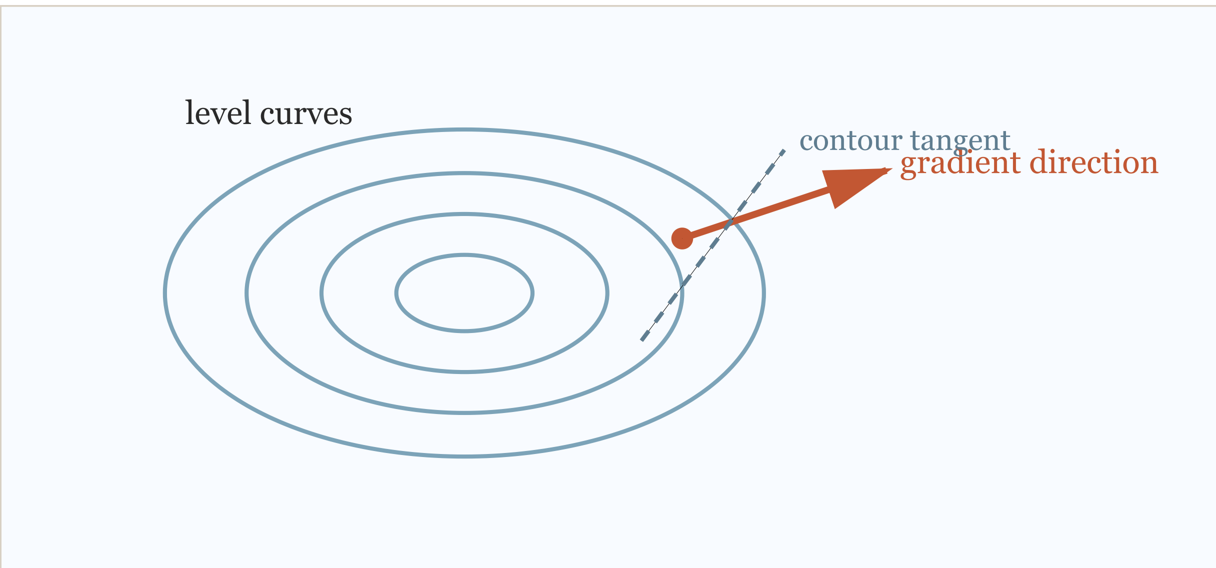
$f(x,y) = x^2 + y^2,$

then

$\nabla f = \langle 2x, 2y \rangle.$

At the point `(1, 2)`, the gradient is $\langle 2, 4 \rangle$, which points directly away from the origin.

That matches the geometry of the bowl surface: the function rises fastest away from the center.



Directional derivative preview

The gradient matters because it controls directional change. If you move in a unit direction `u`, the directional derivative can be read conceptually as the rate at which `f` changes in that direction. The largest possible directional increase occurs when `u` points with the gradient.

Even before formal formulas are introduced, this interpretation helps unify contour maps, steepest ascent, and tangent-plane approximations.

13.5 Tangent planes and linear approximation

In one-variable calculus, a differentiable function is locally approximated by a tangent line.

For a function of two variables, the local linear object is a tangent plane.

At the point `(a, b)`, the linear approximation is

$$L(x,y) = f(a,b) + f_x(a,b)(x - a) + f_y(a,b)(y - b).$$

Example

Let

$$f(x,y) = x^2 + y^2.$$

At `(1, 1)`,

- $f(1,1) = 2$
- $f_x(1,1) = 2$
- $f_y(1,1) = 2$

So the tangent-plane approximation is

$$L(x,y) = 2 + 2(x - 1) + 2(y - 1).$$

This plane gives a good approximation near `(1, 1)`.

Why tangent planes matter

Tangent planes support:

- local estimation,
- sensitivity analysis,
- optimization methods,
- and later multivariable numerical methods.

They are the natural extension of local linearity.

Directional thinking

The gradient and tangent plane together show that multivariable change is directional. Some directions climb steeply, some descend, and some stay level. That richer structure is the main new feature of several-variable calculus.

This is also why contour maps are so powerful: they let the eye detect directional change before any formulas are computed.

A full course usually introduces directional derivatives explicitly. Even before that formula appears, the gradient already hints that change depends on direction, not only on position.

13.6 Directional derivatives and movement in arbitrary directions

Partial derivatives answer two special questions:

- what happens if we move only in the `x` direction?
- what happens if we move only in the `y` direction?

Many real motions do neither. A hiker, airplane, or optimization algorithm usually moves in a direction that mixes both coordinates at once.

If $u = \langle u_1, u_2 \rangle$ is a unit direction vector, then the directional derivative of `f` at `(a, b)` in the direction `u` is

$D_u f(a,b) = \nabla f(a,b) . u.$

This formula says:

- the gradient stores all first-order directional information,
- a chosen direction extracts the component relevant to that motion,
- and the steepest ascent occurs when `u` points with the gradient.

Worked example

Let

$f(x,y) = x^2 + y^2.$

At `(1, 2)`,

$\nabla f = \langle 2, 4 \rangle.$

Take the unit direction

$u = \langle 3/5, 4/5 \rangle.$

Then

$D_u f(1,2) = \langle 2, 4 \rangle . \langle 3/5, 4/5 \rangle = 6/5 + 16/5 = 22/5.$

So moving from `(1, 2)` in that direction increases the function at an instantaneous rate of $22/5$.

Why the gradient appears

If a path is written as

$r(t) = \langle x(t), y(t) \rangle,$

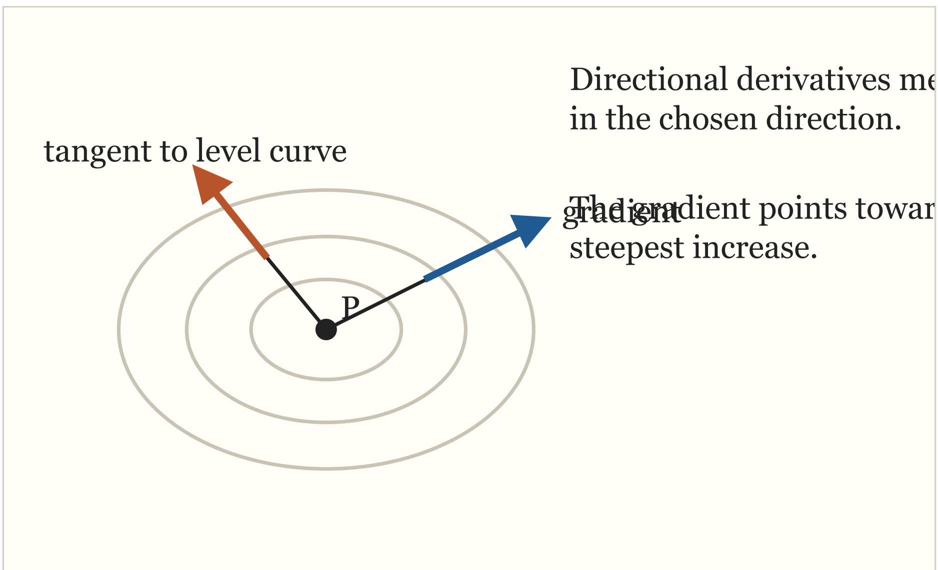
then the chain rule gives

$d/dt f(x(t), y(t)) = f_x x'(t) + f_y y'(t).$

That is exactly the dot product of the gradient with the velocity vector of the path. Directional derivatives are therefore the multivariable chain rule written geometrically.

Tangent directions to level curves

If you move along a level curve, the function value stays constant. So the directional derivative in the tangent direction is `0`. That is why the gradient is perpendicular to level curves: it points across them, not along them.



A classroom habit that prevents mistakes

Always check whether the stated direction is a unit vector. If it is not, normalize it before using the directional-derivative formula.

13.7 Local extrema, critical points, and saddle behavior

In one-variable calculus, a local maximum or minimum often appears where $f'(x) = 0$. In several variables, the corresponding condition is

$\nabla f(a, b) = \langle 0, 0 \rangle.$

Such a point is called a critical point, provided the partial derivatives exist there. As in one variable, not every critical point is an extremum.

Three common outcomes

- local minimum: the surface rises away from the point in nearby directions,
- local maximum: the surface falls away from the point in nearby directions,
- saddle point: the surface rises in some directions and falls in others.

Prototype examples

For

$f(x, y) = x^2 + y^2$,

the origin is a local minimum because nearby points have larger values.

For

$g(x, y) = x^2 - y^2$,

the origin is a saddle point. Moving along the `x` axis increases `g`, while moving along the `y` axis decreases it.

Second-derivative test

If $\nabla f(a, b) = \langle 0, 0 \rangle$ and the second partial derivatives are continuous near `(a, b)`, define

$D = f_x x(a, b) f_y y(a, b) - (f_x y(a, b))^2$.

Then:

- if $D > 0$ and $f_x x(a, b) > 0$, there is a local minimum,
- if $D > 0$ and $f_x x(a, b) < 0$, there is a local maximum,
- if $D < 0$, the point is a saddle,
- if $D = 0$, the test is inconclusive.

This test is a local-shape test for the quadratic part of the surface.

Contour-map interpretation

Contour maps often reveal the classification visually.

- nested closed contours with values increasing outward suggest a minimum,
- nested closed contours with values decreasing outward suggest a maximum,
- contour patterns that cross or open in alternating directions suggest a saddle.

Worked example

Let

$f(x, y) = x^2 + y^2 - 4x + 6y.$

Then

- $f_x = 2x - 4,$
- $f_y = 2y + 6.$

The critical point occurs at `(2, -3)`. Also,

- $f_x x = 2,$
- $f_y y = 2,$
- $f_x y = 0,$

so $D = 4 > 0$ and $f_x x > 0$. Therefore `(2, -3)` is a local minimum.

Why saddle points matter

Optimization algorithms can stall near saddle points because the gradient may be small even though the point is not optimal. This is one reason multivariable calculus appears so often in numerical optimization.

13.8 Constrained optimization and Lagrange multipliers

Many optimization problems are not free. A design may have a fixed budget, a physical system may stay on a surface, or a quantity may be held constant by a conservation law.

Suppose we want to optimize $f(x, y)$ subject to the constraint

$g(x, y) = c.$

At an interior optimum on the constraint curve, the gradients are parallel:

$\nabla f = \lambda \nabla g.$

This is the method of Lagrange multipliers.

Why parallel gradients make sense

The vector ∇g is perpendicular to the constraint curve. If the objective could still increase along the curve, then the point would not be optimal. So at an optimum, the objective's steepest-ascent direction cannot have a tangential component.

Worked example

Maximize

$$f(x,y) = xy$$

subject to

$x + y = 10$, with $x > 0$ and $y > 0$.

Let $g(x,y) = x + y$. Then

- $\nabla f = \langle y, x \rangle$,
- $\nabla g = \langle 1, 1 \rangle$.

Set

$\langle y, x \rangle = \lambda \langle 1, 1 \rangle$.

So $x = y$. Combined with $x + y = 10$, this gives $x = 5$, $y = 5$. The maximum product is therefore `25`.

Boundary awareness

Lagrange multipliers locate candidate points. They do not replace the need to examine endpoints, corners, or domain restrictions when those exist.

Worked examples

Example 1: directional derivative of a temperature field

Let

$T(x,y) = 80 - x^2 - 2y^2$.

At `(2, 1)`,

$\nabla T = \langle -4, -4 \rangle$.

In the northeast direction $u = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$,

$D_u T(2,1) = \langle -4, -4 \rangle . \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle = -4\sqrt{2}$.

So temperature decreases in that direction.

Example 2: tangent-plane estimate

Let

$f(x, y) = \sqrt{25 - x^2 - y^2}$.

At `(3, 0)`,

- $f(3, 0) = 4$,
- $f_x(3, 0) = -3/4$,
- $f_y(3, 0) = 0$.

So the tangent-plane model is

$L(x, y) = 4 - (3/4)(x - 3)$.

Example 3: classifying a critical point

For

$f(x, y) = x^3 + y^3 - 3x - 3y$,

the equations $f_x = 3x^2 - 3 = 0$ and $f_y = 3y^2 - 3 = 0$ give four critical points:

- `(1, 1)`,
- `(1, -1)`,
- `(-1, 1)`,
- `(-1, -1)`.

The second partials are $f_xx = 6x$, $f_yy = 6y$, and $f_xy = 0$. So:

- at `(1, 1)`, there is a local minimum,
- at `(-1, -1)`, there is a local maximum,
- at `(1, -1)` and `(-1, 1)`, there are saddles.

Example 4: constrained maximum on a circle

Maximize

$$f(x, y) = x + 2y$$

subject to

$x^2 + y^2 = 25$.

Here $\nabla f = \langle 1, 2 \rangle$ and $\nabla g = \langle 2x, 2y \rangle$, so the maximizing point lies in the same direction as $\langle 1, 2 \rangle$. Scaling that direction to radius `5` gives

$(x, y) = (\sqrt{5}, 2\sqrt{5})$.

Quick tactics

- Translate contour pictures into words before computing: where is the function steep, flat, or rising fastest?
- For partial derivatives, say out loud which variable moves and which variables are frozen.
- Use path tests to disprove limits quickly, but do not mistake repeated agreement for a proof of existence.
- Treat the tangent plane as the best nearby linear model, not as a globally accurate surface.
- At a critical point, ask whether nearby contour lines close up or cross through.
- For constrained optimization, sketch the constraint first and mark where the objective should plausibly be largest or smallest.

Chapter review

This chapter extends every major theme of single-variable calculus:

- graphs become surfaces,
- tangent lines become tangent planes,
- slopes become partial derivatives,
- uphill direction becomes the gradient,
- and one-dimensional approach behavior becomes path-sensitive multivariable limits.

The biggest conceptual shift is that change is no longer forced to happen along a single axis. Several-variable calculus is therefore a study of local behavior with direction built in.

The toolkit now also includes:

- directional derivatives for arbitrary motion,
- classification of critical points and saddle behavior,
- and constrained optimization by Lagrange multipliers.

Mini projects

Project 1: contour reading portfolio

Collect or create four contour maps from terrain, temperature, economics, or simulated data. For each one, identify steep regions, likely gradient directions, and any plausible local extrema.

Project 2: multivariable approximation note

Choose a function of two variables, compute partial derivatives at a base point, build the tangent-plane approximation, and test the approximation at several nearby points. Report when the linear model is convincing and when it begins to drift.

Common traps

- Treating a multivariable limit as if there were only one way to approach the point.
- Forgetting that a partial derivative freezes the other variables.
- Interpreting the gradient as a point instead of a direction vector.
- Confusing contour lines with the graph itself.
- Using a tangent-plane approximation too far from the base point.
- Forgetting to normalize a direction vector before computing a directional derivative.
- Treating $\nabla f = 0$ as automatic proof of a maximum or minimum.
- Ignoring boundary restrictions in a constrained optimization problem.

Exercises

Warm-up: surfaces, contours, and partial derivatives

1. For $f(x, y) = x^2 + y^2$, describe the level curves.
2. Explain why closely spaced contour lines indicate rapid change.
3. Compute f_x and f_y for $f(x, y) = x^3y + y^2$.
4. Compute f_x and f_y for $f(x, y) = e^x \cos y$.
5. Explain in words what $f_x(a, b)$ measures.
6. Explain in words what $f_y(a, b)$ measures.

Core skill: gradients, tangent planes, and directional derivatives

1. Find ∇f for $f(x, y) = x^2 + 3y^2$.
2. Evaluate the gradient from problem 7 at `(1, -2)`.
3. Find the tangent-plane approximation to $f(x, y) = x^2 + y^2$ at `(1, 2)`.
4. Find the directional derivative of $f(x, y) = x^2 + y^2$ at `(1, 1)` in the direction $\langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$.
5. Find the directional derivative of $f(x, y) = xy$ at `(2, 3)` in the direction $\langle 3/5, 4/5 \rangle$.
6. A particle moves through a temperature field $T(x, y)$. Explain why $\nabla T \cdot v$ gives the instantaneous rate of temperature change along its motion.

Interpretation: limits, continuity, and local shape

1. Show that the limit of $(xy)/(x^2 + y^2)$ at `(0, 0)` does not exist by comparing two paths.
2. Test whether the limit of $(x^2y)/(x^2 + y^2)$ appears path-dependent at `(0, 0)`.
3. Explain why checking only the axes can be misleading in a multivariable limit problem.
4. Describe how a contour map can suggest a local minimum without proving it.
5. Explain the difference between a critical point and an extremum.

Challenge: critical points and classification

1. Find the critical points of $f(x, y) = x^2 + y^2 - 4x + 2y$.
2. Classify the critical point of $f(x, y) = x^2 + y^2 - 4x + 2y$.
3. Find the critical points of $f(x, y) = x^2 - y^2$.
4. Classify the point `(0, 0)` for $f(x, y) = x^2 - y^2$.
5. Find and classify the critical points of $f(x, y) = x^3 - 3x + y^2$.
6. Explain why a saddle point can still satisfy $\nabla f = 0$.

Constrained optimization and modeling

1. Use Lagrange multipliers to maximize `xy` subject to $x + y = 12$.
2. Use Lagrange multipliers to minimize $x^2 + y^2$ subject to $x + y = 6$.
3. Find the points on the circle $x^2 + y^2 = 25$ where $f(x, y) = x - y$ is largest and smallest.
4. A manufacturer uses fixed material for the side lengths `x` and `y` of a rectangular panel with constraint $2x + 3y = 60$. Which quantity would you optimize if you wanted maximum area?
5. Explain why endpoints and corners still matter even when Lagrange multipliers are used.

Challenge and synthesis

1. Give a physical interpretation of the gradient for an elevation surface.
2. Explain why the tangent plane is the multivariable version of a tangent line.
3. Describe how contour maps, gradients, and directional derivatives all express the same local geometry in different languages.
4. A function has $\nabla f(a, b) = \langle 0, 0 \rangle$ and $D = 0$ in the second-derivative test. Explain what additional evidence you would seek.

Proof window: why different paths matter

In one variable, there are only two basic directions of approach. In two variables, there are infinitely many curves leading to a point. If a proposed limit changes with the path, then there is no single nearby behavior for the function to settle into.

That is why path testing is such a powerful first tool for showing a multivariable limit does not exist.

Reflection

Multivariable calculus keeps the original logic of calculus intact: study local change, then use that local information to understand global structure. The difference is that change can now happen in many directions at once.

Chapter 14. Multiple Integration

Opening question

How do we accumulate over a region rather than along a line?

If a density covers a sheet of metal, or a temperature field covers a floor plan, then totals are built over two-dimensional or three-dimensional regions. Multiple integration extends the accumulation idea to those settings.

Learning goals

By the end of this chapter, you should be able to:

- interpret double and triple integrals as accumulated totals over regions,
- set up double integrals over rectangles and simple regions,
- explain when polar coordinates are useful,
- understand the idea of triple integration,
- and recognize change of variables as a geometric re-encoding of a region.

Preview questions

- How does a double integral differ conceptually from an iterated integral?
- Why do coordinate changes alter the tiny area or volume element?
- When is changing the order of integration worth the effort?

14.1 Double integrals over rectangles

If $f(x, y)$ gives density or height over a rectangle `R`, then the double integral

$$\iint_R f(x, y)\,dA$$

adds contributions from tiny area pieces.

Small piece interpretation

Over a tiny rectangle,

$\text{small contribution} \approx f(x, y)\Delta A.$

Adding all such pieces and refining the partition leads to the double integral.

Example: constant density

If $f(x, y) = 3$ on the rectangle $0 \le x \le 2, 0 \le y \le 4$, then

$\iint_R 3\,dA = 3(\text{area of } R) = 3(8) = 24.$

This agrees with the constant-function logic from one-variable integration.

Average value over a region

Just as one-variable integrals support average value on an interval, double integrals support average value over a region:

$\text{average} = (1/\text{area}(R)) \iint_R f\,dA.$

This is useful when the function describes temperature, density, or any distributed quantity.

Height interpretation

If $f(x, y)$ is nonnegative, the double integral can also be viewed as the volume under the surface $z = f(x, y)$ above the region `R`. This interpretation often helps students connect multiple integration with the earlier single-variable volume ideas.

14.2 Double integrals over general regions

Not every region is a rectangle.

For more general regions, we often describe the boundaries with inequalities and then write the integral in iterated form.

Example

If the region is bounded by

- $0 \le x \le 1$
- $0 \le y \le x$

then

$$\int_0^1 \int_0^x f(x,y)\,dy\,dx$$

integrates over the triangular region.

Why iterated integrals help

An iterated integral turns a two-dimensional accumulation into two one-dimensional accumulations performed in sequence.

This is not merely a computational trick. It reflects the idea that a region can be swept out one slice at a time.

In many common settings, the order of integration can be reversed if the region is described correctly. Choosing the more natural order often makes the integral far easier to evaluate.

Worked example: reversing the order

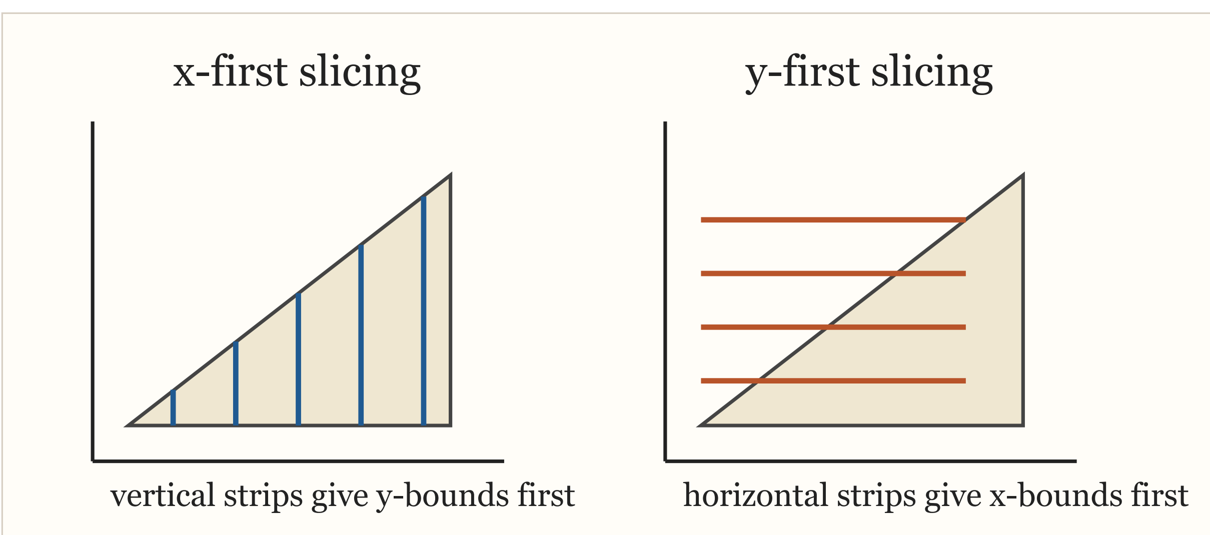
Consider

$\int_0^1 \int_x^1 f(x,y)\,dy\,dx.$

The region consists of points satisfying $0 \le x \le 1$ and $x \le y \le 1$. Rewriting in the opposite order gives

$\int_0^1 \int_0^y f(x,y)\,dx\,dy.$

The geometry matters more than the symbols. A quick sketch reveals which variable should sweep the region first.



Order-of-integration strategy

When deciding whether to reverse the order of integration, ask:

1. Which variable has simpler bounds?
2. Which order avoids splitting the region into pieces?
3. Which order matches the easiest inner integral?

These are practical questions, but they reflect real geometric insight. A region can often be swept in more than one direction; the best order is the one that keeps the sweep simplest.

14.3 Polar coordinates

Some regions are awkward in rectangular coordinates but simple in polar coordinates.

In polar form:

- $x = r\cos\theta$
- $y = r\sin\theta$

and the area element becomes

$dA = r dr dtheta.$

Why the extra `r` appears

A small polar region is approximately a thin sector. Its area depends on both radial thickness and distance from the origin. Farther from the origin, the same change in angle sweeps out a larger arc.

Example: a disk

For the disk of radius `a`,

$0 \leq r \leq a, 0 \leq \theta \leq 2pi.$

So the area is

$\int_0^{2pi} \int_0^a r dr dtheta = \pi a^2.$

Polar coordinates are useful whenever circles, disks, radial symmetry, or angle-based boundaries appear naturally.

Beyond disks

Polar coordinates also simplify sectors, annuli, spirals, and regions bounded by curves written naturally as $r = g(\theta)$.

Symmetry is a major clue

Whenever a region or integrand is built around distance from the origin, circles, or rotational symmetry, polar coordinates deserve immediate consideration. The coordinate system should match the geometry whenever possible.

14.4 Triple integrals

A triple integral extends the same idea to three-dimensional accumulation.

If $\varrho(x, y, z)$ is a density over a solid `D`, then

$$\iiint_D \varrho \, dV$$

gives total mass.

Interpretation

The triple integral is built from tiny volume pieces. Each contributes approximately

$\varrho \Delta V$.

This is the three-dimensional version of the same accumulation logic used throughout the book.

Example

If $\varrho = 2$ on a box of side lengths `1`, `2`, and `3`, then the total mass is

$2(1)(2)(3) = 12$.

Triple integrals and physical interpretation

Triple integrals commonly represent:

- mass from volume density,
- total heat or charge in a region,
- average values over solids,
- and probabilities in multivariable settings.

The computation may still be technical, but the underlying meaning is unchanged: each small volume contributes a local amount, and the integral accumulates those contributions.

14.5 Change of variables

Sometimes a region or integrand becomes simpler after a coordinate transformation.

Change of variables re-expresses the integral in a new coordinate system and corrects the area or volume element accordingly.

Polar coordinates are the first major example. Later examples include more general linear or nonlinear transformations.

Geometric meaning

Changing variables is not just symbol substitution. It changes how pieces of area or volume are measured. The correction factor accounts for the local stretching or compression of the transformation.

This matters because a transformation can turn a difficult region into a simple rectangle in the new coordinates while quietly distorting area. The correction factor repairs that distortion.

That point is easy to miss if change of variables is taught as symbol pushing. Geometrically, it is really a change in how tiny pieces are measured.

Jacobian intuition without heavy formalism

In a full course, the correction factor is encoded by a Jacobian determinant. Even before that formalism is developed, the idea is simple: a transformation can stretch, compress, shear, or rotate small pieces. The correction factor records how much the size of those pieces changes.

Polar coordinates provide the cleanest example. A tiny wedge farther from the origin is wider than a tiny wedge near the center, so the area factor must depend on `r`.

14.6 Mass, moments, and centers of mass

One of the main reasons to study double and triple integrals is that they compute physical totals over distributed matter.

If a lamina occupying a region `R` has density $\varrho(x, y)$, then its mass is

$$m = \iint_R \varrho(x, y)\, dA.$$

Moments

To locate balance, we measure how mass is distributed relative to the coordinate axes:

- $M_x = \iint_R y\varrho(x,y)dA$,
- $M_y = \iint_R x\varrho(x,y)dA$.

The center of mass is then

- $xbar = M_y/m$,
- $ybar = M_x/m$.

These formulas are weighted averages. More mass farther from an axis creates a larger moment.

Worked example

Let `R` be the rectangle $0 \le x \le 2, 0 \le y \le 1$, and let $\varrho(x,y) = 1 + x$.

Then

$m = \int_0^2 \int_0^1 (1 + x)dydx = \int_0^2 (1 + x)dx = 4.$

Also

$M_y = \int_0^2 \int_0^1 x(1 + x)dydx = \int_0^2 (x + x^2)dx = 14/3.$

So

$xbar = M_y/m = 7/6.$

The balance point moves rightward because density increases with `x`.

Average value revisited

Average value over a region is another weighted-accumulation idea:

$f_avg = (1/area(R)) \iint_R f\,dA.$

In applications, averages over regions model average temperature, average pollutant concentration, and average stress over a surface.

14.7 Cylindrical and spherical coordinates

When three-dimensional geometry has circular or radial symmetry, rectangular coordinates often hide the structure of the problem.

Cylindrical coordinates

Cylindrical coordinates extend polar coordinates into space:

- $x = r\cos\theta$,
- $y = r\sin\theta$,
- $z = z$,
- $dV = r dr d theta dz$.

This system is ideal for cylinders, tubes, and solids formed around the `z` axis.

Spherical coordinates

Spherical coordinates describe points by distance from the origin and two angles:

- $x = \varrho \sin\phi \cos\theta$,
- $y = \varrho \sin\phi \sin\theta$,
- $z = \varrho \cos\phi$,
- $dV = \varrho^2 \sin\phi d\varrho d\phi d\theta$.

The factor $\varrho^2 \sin\phi$ is the three-dimensional analogue of the `r` factor in polar coordinates.

Worked example: volume of a sphere

For the sphere of radius `a`,

- $0 \le \varrho \le a$,
- $0 \le \phi \le \pi$,
- $0 \le \theta \le 2pi$.

So

$$V = \int_0^{2pi} \int_0^{\pi} \int_0^{a} \varrho^2 \sin\phi d\varrho d\phi d\theta = 4pi a^3 / 3.$$

Choosing between cylindrical and spherical coordinates

Use cylindrical coordinates when the geometry is circular around an axis. Use spherical coordinates when the geometry is built around distance from a center.

14.8 Probability densities and accumulated expectation

Multiple integrals also appear naturally in probability.

If $p(x, y)$ is a probability density over a region `R`, then

$$\iint_R p(x, y)\,dA = 1.$$

Expected value

An expected value is a weighted average. For example,

$$E[x] = \iint_R x\,p(x, y)\,dA.$$

This is mathematically the same structure as a moment divided by total mass, except the total mass has been normalized to `1`.

Why probability belongs here

Probability densities give students a second interpretation of double and triple integrals beyond area and mass. The same accumulated-total framework works in both settings.

Quick tactics

- Sketch the region before writing any bounds.
- If one order of integration creates messy limits, test the reversed order.
- When the region is circular or radial, ask about polar coordinates before doing harder algebra.
- Keep the meaning of `dA` or `dV` visible; it is part of the geometry, not a symbolic leftover.
- For mass and center-of-mass problems, decide first whether the density is constant or varying.
- In three dimensions, let the symmetry choose the coordinate system whenever possible.

Chapter review

Multiple integration extends the book's accumulation logic from intervals to regions and solids. The key ideas are:

- build the total from tiny area or volume pieces,
- organize the accumulation through iterated integrals,
- match the coordinate system to the geometry,
- and correct for any distortion created by a coordinate change.

This chapter is where calculus begins to feel fully geometric. The local piece is still the organizing unit, but now the space being accumulated over may curve, rotate, or stretch.

The richer applications include:

- mass and balance of nonuniform objects,
- coordinate systems adapted to cylinders and spheres,
- and probability densities over planar or spatial regions.

Mini projects

Project 1: same region, two orders

Choose a nonrectangular planar region and write the corresponding double integral in both orders. Evaluate both if possible, and explain which order is more natural and why.

Project 2: coordinate-system comparison

Choose a radially symmetric region and compare a rectangular-coordinate setup with a polar-coordinate setup. Explain how the geometry changes the bounds, the differential element, and the overall complexity.

Project 3: center of mass design memo

Model a thin plate with nonconstant density, compute its mass and center of mass, and explain how the density choice shifts the balance point.

Common traps

- Forgetting that `dA` or `dV` carries geometric meaning.
- Setting incorrect bounds for a nonrectangular region.
- Dropping the extra `r` in polar coordinates.
- Treating iterated integrals as algebra only rather than slice-by-slice accumulation.
- Forgetting that coordinate changes distort area and volume.
- Confusing a mass integral with an average-value calculation.
- Using spherical coordinates when cylindrical symmetry is really the dominant feature.

Proof window: why $dA = r dr dtheta$ is plausible

A tiny polar sector has one side of length approximately `dr` and the other of length approximately `r dtheta`. Multiplying those gives an area element of about

`r dr dtheta`.

That is the local geometric reason behind the polar area factor.

Exercises

Warm-up: double and triple integral meaning

1. What kind of quantity does a double integral accumulate?
2. When are polar coordinates especially useful?
3. Why does a triple integral naturally model mass?

Core skill: iterated integrals and polar setup

1. Compute $\int_0^2 \int_0^3 1 dy dx$.
2. Write an iterated integral for the triangular region $0 \le y \le x \le 1$.
3. Find the area of a disk of radius `2` using polar coordinates.

Interpretation: geometry of repeated integration

1. Explain why a double integral can be seen as repeated one-variable integration.
2. Explain the geometric meaning of the factor `r` in polar coordinates.

Challenge: coordinate choice and correction factors

1. Describe a region where rectangular coordinates are awkward but polar coordinates are natural.
2. Explain why change of variables needs a correction factor.
3. A density is larger near the outer edge of a disk than near the center. Why might polar coordinates be a natural choice?

Modeling: mass and concentration

1. A thin plate has density $\rho(x, y)$ over a rectangular sheet. Explain how a double integral would compute its total mass.
2. A pollutant concentration $c(x, y, z)$ fills a room. Explain how a triple integral would estimate the total pollutant amount.

Mass, moments, and centers

1. Compute the mass of the rectangle $0 \le x \le 2, 0 \le y \le 1$ with density $\rho(x, y) = 3$.
2. Compute the mass of the same rectangle with density $\rho(x, y) = 1 + x$.
3. Explain why increasing density to the right should move the center of mass rightward.
4. State formulas for M_x, M_y, and the center of mass of a lamina.

5. A plate has density proportional to distance from the `y` axis. Which moment would you expect to change most noticeably?

Cylindrical and spherical coordinates

1. Write the cylindrical-coordinate relations between `(x, y, z)` and (r, θ, z).
2. Write the spherical-coordinate relations between `(x, y, z)` and (ϱ, ϕ, θ).
3. Explain why $dV = r dr dthetadz$ in cylindrical coordinates.
4. Explain why spherical coordinates are natural for a ball centered at the origin.
5. Set up, but do not evaluate, the volume integral for a sphere of radius `4` in spherical coordinates.

Probability and synthesis

1. Explain why a probability density must integrate to `1`.
2. Compare average value over a region with expected value for a probability density.
3. Describe a real context in which a triple integral measures accumulated heat or pollutant concentration in a room.

Reflection

Multiple integration shows that accumulation is not tied to a line. The central calculus idea survives in higher dimensions: describe a small local contribution, then add it over the whole region.

Chapter 15. Vector Calculus

Opening question

What does it mean to accumulate along a path or across a surface instead of over a plain interval or region?

Vector calculus answers that question. It studies vector fields, circulation, flux, and the great theorems that connect local behavior to global accumulation.

Learning goals

By the end of this chapter, you should be able to:

- interpret vector fields,
- understand line integrals at a conceptual level,
- explain the ideas behind Green's Theorem,
- understand the meaning of surface integrals and flux,
- and describe the roles of divergence and Stokes' Theorem.

Preview questions

- Why can a field push strongly while still doing little work along a particular path?
- What is the difference between moving along a boundary and measuring what crosses through it?
- How are the major integral theorems versions of one shared boundary-interior principle?

15.1 Vector fields

A vector field assigns a vector to each point in space.

Examples:

- a velocity field in a fluid,
- a force field,
- an electric field,

- a gradient field of steepest increase.

Example

$$F(x, y) = \langle -y, x \rangle$$

is a rotational field in the plane. The vectors circle around the origin.

Vector fields turn geometry and physics into a calculus setting where direction matters at every point.

Conservative intuition

Some vector fields behave like gradients of potential functions. In those fields, path independence often appears, and line integrals depend only on endpoints. This is one of the major organizing ideas behind vector calculus.

That makes conservative fields the natural meeting point between vector geometry and the antiderivative logic from earlier chapters.

Two kinds of fields to compare

It is helpful to compare two prototype fields:

- radial fields, which point away from or toward a center,
- rotational fields, which circulate around a center.

Radial fields often suggest flux questions. Rotational fields often suggest circulation questions. Many real vector fields contain both features, but separating the prototypes helps build intuition.

15.2 Line integrals

A line integral accumulates along a curve.

There are two common ideas:

- accumulate a scalar quantity along arc length,
- accumulate a vector field along a path.

Work interpretation

If `F` is a force field and $r(t)$ is a path, then the line integral of `F` along the path measures work.

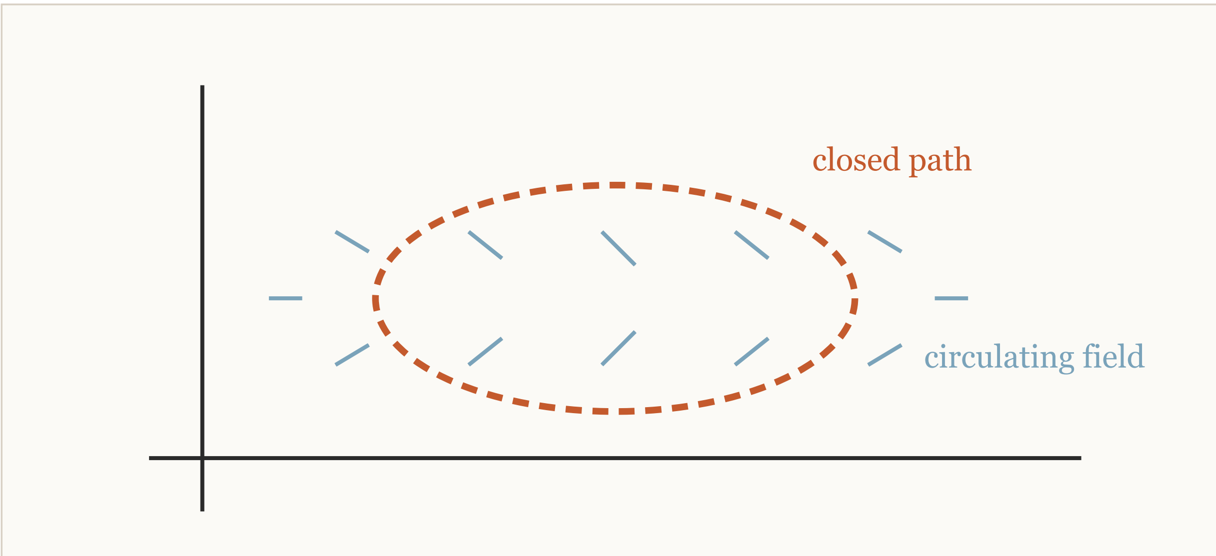
The key local piece is:

`force component along the path * small displacement`.

This is the path-based analogue of the slice methods seen earlier.

Parameterization matters

A path must be described clearly, because the line integral depends on how the field interacts with the path geometry.



Changing the path can change the accumulated work, even if the endpoints stay the same. That is one reason geometry matters so much more in vector calculus than in many one-variable problems.

Orientation matters too

A curve is not just a geometric trace; it also has a direction. Reversing orientation changes the sign of many line integrals because the local displacement vector reverses. This is the vector-calculus version of reversing the bounds in a definite integral.

15.3 Green's Theorem

Green's Theorem connects a line integral around a closed curve in the plane to a double integral over the region inside.

If `C` is a positively oriented, piecewise smooth, simple closed curve bounding a region `R`, and if `P` and `Q` have continuous partial derivatives on an open set containing `R`, then

$$\oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA.$$

Conceptually, it says:

> boundary circulation can be computed from interior rotational behavior.

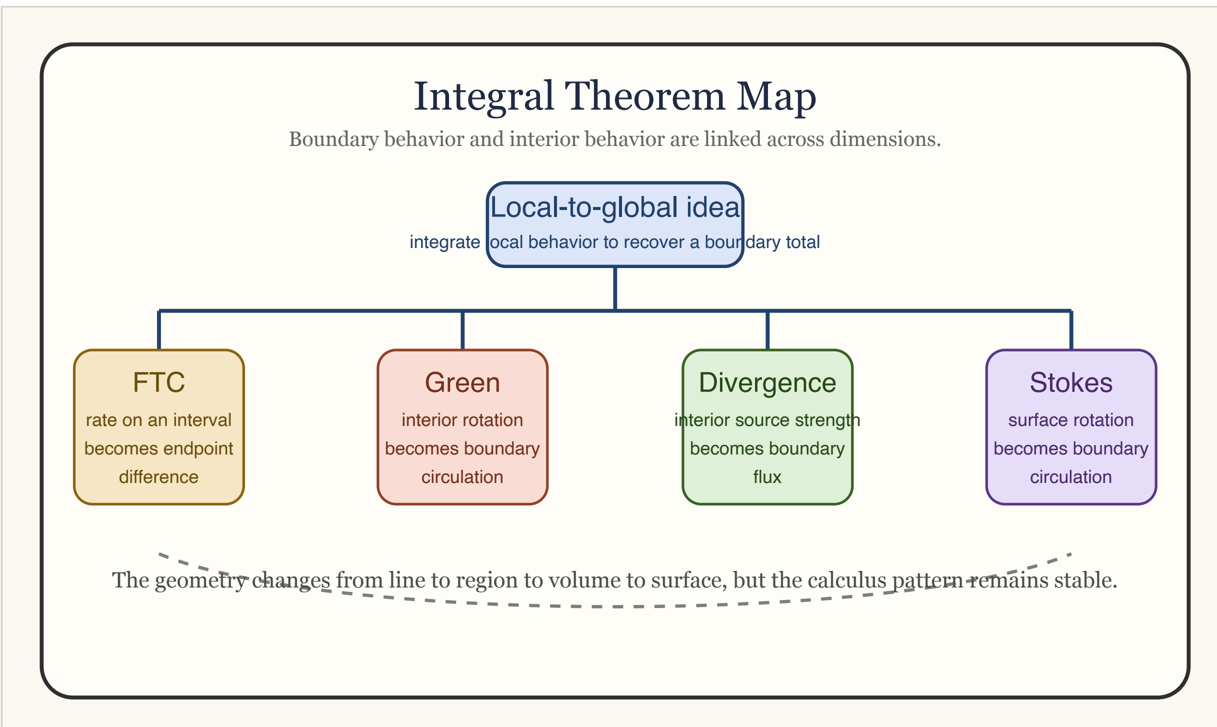
This is a powerful local-to-global principle:

- local rotation inside,
- total circulation on the boundary.

Green's Theorem is one of the earliest places where vector calculus reveals a deep unifying structure.

A boundary-first viewpoint

Green's Theorem is especially useful because sometimes the boundary integral is easier to compute, while in other problems the interior double integral is simpler. The theorem allows either viewpoint.



Curl intuition

Green's Theorem is easiest to remember if the interior quantity is read as local turning or circulation density. In that language, the theorem says that the total turning tendency spread across the region is reflected in the circulation around its edge.

15.4 Surface integrals

A surface integral accumulates over a surface rather than along a line or over a flat region.

When a vector field passes through a surface, the natural quantity is flux: how much of the field crosses the surface.

Flux idea

The local contribution depends on:

- the strength of the field,
- the area of the tiny surface patch,
- the angle between the field and the surface orientation.

If the field runs tangent to the surface, the flux is small. If it points straight through, the flux is large.

This is the surface version of projecting a vector onto the relevant direction before accumulating.

Surface orientation

Just as curves have orientation, surfaces have orientation as well. Choosing the "positive" normal direction determines the sign convention for flux. Reversing the chosen normal flips the sign of the flux integral.

15.5 Divergence and Stokes' Theorem

Vector calculus has several major local-to-global theorems.

Divergence

Divergence measures local source or sink behavior of a vector field.

- positive divergence suggests net outward flow,
- negative divergence suggests net inward flow.

If `S` is a closed, piecewise smooth surface oriented by the outward normal and `F` has continuous first partial derivatives on the enclosed volume `E`, then

$$\iint_S F \cdot n \, dS = \iiint_E \operatorname{div} F \, dV.$$

The Divergence Theorem connects that local source strength inside a volume to total outward flux across the boundary surface.

Stokes' Theorem

Stokes' Theorem connects circulation around the boundary of a surface to rotational behavior across the surface.

If `S` is an oriented, piecewise smooth surface with positively oriented boundary curve `C`, and `F` has continuous first partial derivatives on a region containing `S`, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, dS.$$

It is the three-dimensional analogue of Green's Theorem.

Why these theorems matter

These theorems are not random formulas. They are culminating expressions of a central calculus idea:

> local differential behavior and global accumulated behavior are deeply linked.

That is the same philosophy that powered the Fundamental Theorem of Calculus, now lifted into higher dimensions and richer geometry.

Computational scope

This chapter gives a first computational and conceptual treatment rather than the full technical density of a dedicated vector-calculus course. The emphasis is on correct setup, theorem choice, orientation, and meaning, so that later computational practice rests on a coherent structure.

Seen this way, vector calculus is less a collection of advanced tricks than a final unification of the book's main themes: local rates, geometric structure, and global accumulation.

15.6 Conservative fields and potential functions

Some vector fields are gradients of scalar functions:

$$\mathbf{F} = \nabla f.$$

These are called conservative fields. They matter because line integrals in such fields often depend only on the endpoints, not on the path itself.

Why "conservative" is a good name

If a force field is conservative, the work done around a closed loop is `0`. No net energy is gained by going around the loop and returning to the starting point.

Prototype example

Let

$f(x, y) = x^2 + y^2$.

Then

$\nabla f = \langle 2x, 2y \rangle$.

So the field $F(x, y) = \langle 2x, 2y \rangle$ is conservative. A line integral of `F` from point `A` to point `B` equals $f(B) - f(A)$.

Why potentials simplify work problems

When a potential function exists, a difficult path integral can collapse to endpoint evaluation. This is the vector-calculus echo of antidifferentiation in one-variable calculus.

Warning sign

Not every field is conservative. Rotational fields such as $F(x, y) = \langle -y, x \rangle$ usually produce nonzero circulation around closed curves.

15.7 Parametrized surfaces and normal vectors

To integrate over a surface, we need a way to describe the surface and a way to determine which direction counts as positive crossing.

A parametrized surface has the form

$r(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$.

The tangent vectors r_u and r_v lie in the surface. Their cross product

$$r_u \times r_v$$

is normal to the surface.

Why normals matter

Flux depends on how strongly the field points through the surface. That requires a normal direction, not a tangent direction.

Worked example

For the plane

$$z = 2x + y$$

above a parameter rectangle in the `xy` plane, one convenient parameterization is

$r(u, v) = \langle u, v, 2u + v \rangle.$

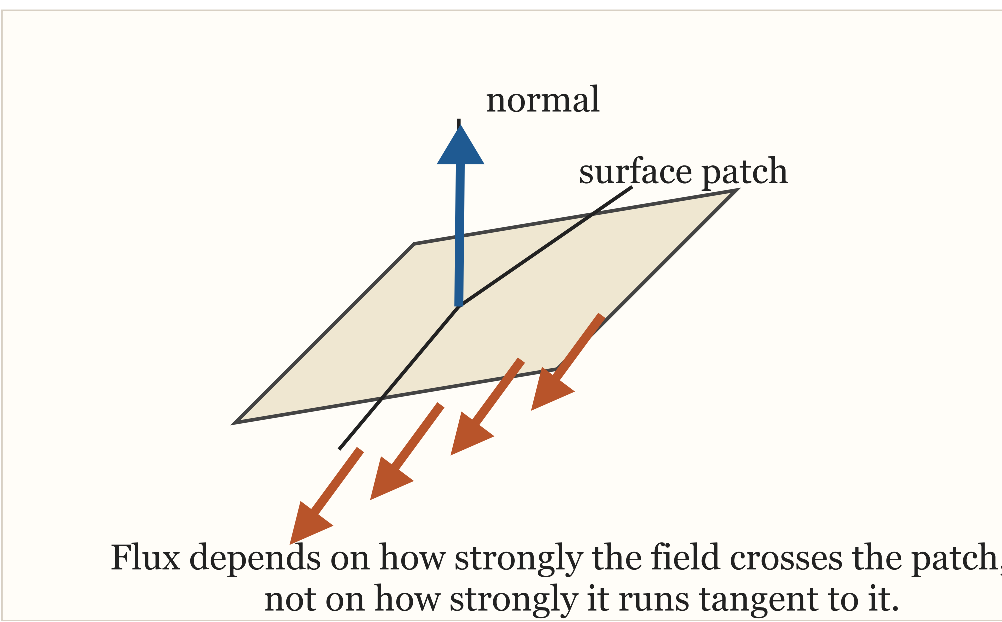
Then

- $r_u = \langle 1, 0, 2 \rangle$,
- $r_v = \langle 0, 1, 1 \rangle$,
- $r_u \times r_v = \langle -2, -1, 1 \rangle$.

That vector provides an oriented normal for the surface patch.

Surface area element

The magnitude $|r_u \times r_v|$ measures how a tiny parameter rectangle stretches into a tiny surface patch. This is the surface version of the Jacobian idea from change of variables.



15.8 Choosing the right theorem

In a full vector-calculus course, students often know several powerful theorems but struggle to decide which one to apply. A short decision framework is more useful than memorizing isolated slogans.

Green's Theorem is natural when

- the field is planar,
- the curve is closed,
- and the problem compares boundary circulation with interior rotation.

The Divergence Theorem is natural when

- the quantity of interest is outward flux,
- the boundary encloses a volume,
- and a volume integral is easier than a direct surface computation.

Stokes' Theorem is natural when

- the problem involves circulation around a space curve,
- that curve is the boundary of an oriented surface,
- and the field's rotational tendency across the surface is easier to compute.

A practical checklist

Ask these questions in order:

1. Is the accumulation along a curve, across a surface, or through a closed boundary?
2. Is the region planar or spatial?
3. Would a boundary computation or an interior computation be simpler?
4. Is the field likely conservative?

These questions quickly narrow the appropriate theorem.

Worked examples

Example 1: work in a conservative field

Let

$F(x, y) = \langle 2x, 2y \rangle.$

A potential is $f(x,y) = x^2 + y^2$. So the work from `(1, 0)` to `(2, 3)` is

$f(2,3) - f(1,0) = (4 + 9) - 1 = 12.$

Example 2: circulation in a rotational field

For

$F(x,y) = \langle -y, x \rangle,$

the field swirls around the origin. A closed counterclockwise path around the origin naturally has positive circulation because the field tends to push with the motion rather than across it.

Example 3: flux through a horizontal surface

Suppose

$F(x,y,z) = \langle 0,0,5 \rangle.$

Across any horizontal patch with upward normal, the flux is positive because the field points straight through the surface. Across a vertical wall, the flux is `0` because the field is tangent to that wall.

Example 4: theorem selection

If a problem asks for outward flux through a closed sphere in a field whose divergence is easy to compute, the Divergence Theorem should be tested before attempting a direct surface integral. If a problem asks for circulation around a closed planar curve, Green's Theorem may be the correct compression.

Example 5: Green's Theorem on the unit square

Let

$$F(x,y) = \langle 0, x^2 \rangle$$

and let `C` be the positively oriented boundary of the unit square.

Here $P = 0$ and $Q = x^2$, so

$Q_x - P_y = 2x.$

Green's Theorem gives

$\oint_C P\,dx + Q\,dy = \int_0^1 \int_0^1 2x\,dA.$

Evaluating,

$\int_0^1 \int_0^1 2x\,dy\,dx = \int_0^1 2x\,dx = 1.$

This example is useful because the theorem turns a four-segment boundary calculation into one compact interior integral.

Example 6: Divergence Theorem on a cube

Let

$$F(x, y, z) = \langle x, y, z \rangle$$

and let `S` be the boundary of the unit cube $0 \leq x, y, z \leq 1$.

Since

$\operatorname{div} F = 1 + 1 + 1 = 3,$

the Divergence Theorem gives

$\iint_S F \cdot n\,dS = \iiint_E 3\,dV = 3.$

The geometry is the whole point. The theorem avoids computing flux separately on all six faces.

Example 7: Stokes' Theorem on a flat disk

Let

$$F(x, y, z) = \langle -y, x, 0 \rangle$$

and let `C` be the unit circle in the plane $z = 0$, oriented counterclockwise as viewed from above.

The curl is

$\operatorname{curl} F = \langle 0, 0, 2 \rangle.$

Choose the flat unit disk as the spanning surface with upward normal $n = \langle 0, 0, 1 \rangle$. Then

$\operatorname{curl} F \cdot n = 2,$

so

$\oint_C F \cdot dr = \iint_S \operatorname{curl} F \cdot n\,dS = \iint_S 2\,dS = 2pi.$

This is one of the cleanest demonstrations that Stokes' Theorem is a circulation theorem rather than a surface-area theorem.

Hypothesis checklist for the big theorems

Before using Green's, Divergence, or Stokes, pause and check:

- Is the curve or surface oriented?
- Is the boundary closed when the theorem requires closure?
- Are the field components differentiable on a region containing the geometry?
- Are you matching the theorem to the quantity actually requested: work, circulation, or flux?

These checks are short, but they are where many classroom errors begin.

Quick tactics

- Identify whether the problem is about motion along a path or flow across a boundary.
- Sketch the field and the path or surface together before deciding which quantity is being accumulated.
- Check orientation explicitly; sign errors in vector calculus often begin there.
- When a field looks like a gradient, ask whether path independence may simplify the problem.
- If the surface is parametrized, compute tangent vectors before thinking about flux.
- Choose the theorem that lowers the dimensional difficulty whenever geometry allows it.

Chapter review

Vector calculus is the geometric culmination of the book. Earlier chapters studied change and accumulation on intervals, then on regions and solids. This chapter adds direction, orientation, and boundary geometry.

The major ideas form a compact hierarchy:

- vector fields assign directions and magnitudes through space,
- line and surface integrals accumulate along paths and across surfaces,
- circulation and flux capture different physical questions,
- and the large theorems connect interior differential behavior with boundary totals.

Seeing those ideas as one family is more important than memorizing every formula in isolation.

The extended toolkit also includes:

- conservative fields and potential functions,
- parametrized surfaces and oriented normals,

- and theorem selection based on the geometry of the problem.

Mini projects

Project 1: field atlas

Create a small atlas of vector fields including at least one radial field, one rotational field, and one gradient field. For each field, describe what line-integral or flux questions would be natural and why.

Project 2: theorem comparison essay

Write a short essay comparing the Fundamental Theorem of Calculus, Green's Theorem, the Divergence Theorem, and Stokes' Theorem. Focus on the common boundary-versus-interior structure and on the geometric object integrated in each case.

Common traps

- Treating a vector field like a scalar field.
- Forgetting that path and surface orientation matter.
- Confusing flux with circulation.
- Thinking the big theorems are unrelated, rather than parallel versions of one local-to-global theme.
- Ignoring the geometric meaning of the integrand.
- Forgetting that a closed-loop integral in a conservative field should be `0`.
- Choosing a theorem by name recognition rather than by the shape of the region and the quantity being measured.

Proof window: the recurring pattern behind the big theorems

The Fundamental Theorem of Calculus, Green's Theorem, the Divergence Theorem, and Stokes' Theorem all share one structural pattern:

- measure a local rate-like quantity inside,
- integrate it over a region,
- recover a global accumulated quantity on the boundary.

The geometry changes from theorem to theorem, but the underlying calculus philosophy stays remarkably stable.

Exercises

Warm-up: vector fields and integral meanings

1. What is a vector field?
2. What physical quantity does a work line integral represent?
3. What does flux measure?

Core skill: describing fields and big-theorem ideas

1. Describe the field $F(x, y) = \langle x, y \rangle$.
2. Describe the field $F(x, y) = \langle -y, x \rangle$.
3. Explain in words the difference between circulation and flux.

Interpretation: circulation, flux, and boundary thinking

1. Explain why a line integral is a path-based accumulation.
2. Explain how Green's Theorem resembles the Fundamental Theorem of Calculus.

Challenge: orientation and theorem philosophy

1. Give an example of a situation where boundary behavior is easier to measure than interior behavior.
2. Explain why orientation matters for surface flux.
3. Describe the common philosophical pattern shared by the major integral theorems of calculus.

Modeling: divergence and work

1. A fluid flows through a pipe network. Explain what a positive divergence at a point would mean physically.
2. A charged particle moves through an electric field along a path. Explain how a line integral models the work done.

Conservative fields and potentials

1. Show that $F(x, y) = \langle 2x, 2y \rangle$ is conservative by finding a potential.
2. Find a potential function for $F(x, y) = \langle 3x^2, 4y \rangle$.
3. Explain why a conservative field has path-independent work.
4. Give an example of a field that looks rotational rather than conservative.

Surface geometry and theorem choice

1. Explain why a normal vector is needed for flux.
2. For $r(u, v) = \langle u, v, u + v \rangle$, compute r_u, r_v, and $r_u \times r_v$.
3. Explain why $|r_u \times r_v|$ behaves like a surface-area correction factor.
4. Describe a problem for which Green's Theorem would be more efficient than a direct line integral.
5. Describe a problem for which the Divergence Theorem would be more efficient than a direct surface integral.
6. Describe a problem for which Stokes' Theorem would be more efficient than a direct circulation computation.

Synthesis

1. Compare the Fundamental Theorem of Calculus with Green's Theorem in one paragraph.
2. Compare Green's Theorem with Stokes' Theorem in one paragraph.
3. Explain how conservative fields connect vector calculus to the antiderivative idea from earlier chapters.

Reflection

Vector calculus completes the long arc of the subject. What began as rate and accumulation on a line becomes circulation, flux, and geometry in higher dimensions, but the central message does not change: local behavior and global totals are tied together by calculus.

Chapter 16. Parametric Curves, Polar Coordinates, and Curvature

Opening question

How do we do calculus on curves that double back, loop, or revolve around a point?

Many important curves are awkward or impossible to describe as graphs $y = f(x)$. A circle fails the vertical-line test. A loop may pass through the same point twice. An orbit is often more naturally described by angle and distance than by rectangular coordinates. Mainstream calculus texts therefore devote a substantial chapter to planar parametric curves and polar coordinates. This chapter supplies that missing treatment and connects it to curvature, motion, and geometric approximation.

Learning goals

By the end of this chapter, you should be able to:

- describe a plane curve parametrically and interpret orientation,
- compute slopes, tangent lines, and second-derivative information for parametric curves,
- use arc-length and area formulas in parametric and polar form,
- graph and interpret curves in polar coordinates,
- explain why the factor `r` appears in polar area,
- and interpret curvature as a quantitative measure of turning.

Preview questions

- Why is a parameter often more natural than solving directly for `y` in terms of `x`?
- What geometric information is lost when we eliminate the parameter too early?
- Why does polar area involve r^2, while polar arc length involves both `r` and `dr/dtheta`?
- What does curvature measure that slope alone cannot?

16.1 Parametric curves in the plane

A parametric curve in the plane is given by coordinate functions

$$x = x(t), y = y(t),$$

or in vector form,

$$r(t) = \langle x(t), y(t) \rangle.$$

The parameter `t` often represents time, but it may simply be a running label. The central idea is that the curve is produced by motion: as `t` changes, the point $(x(t), y(t))$ moves through the plane.

Why parametric description matters

Parametric descriptions are natural when:

- the curve represents motion,
- the same `x`-value corresponds to more than one `y`-value,
- a path crosses itself,
- or the geometry is generated by rotation, vibration, or orbit.

Parametrization keeps track not only of which points lie on a curve, but also:

- the order in which those points are traced,
- the speed of tracing,
- and the direction of travel.

If we eliminate the parameter immediately, that dynamical information disappears.

Example: a circle traced once

Consider

$$x = \cos t, y = \sin t, 0 \leq t \leq 2pi.$$

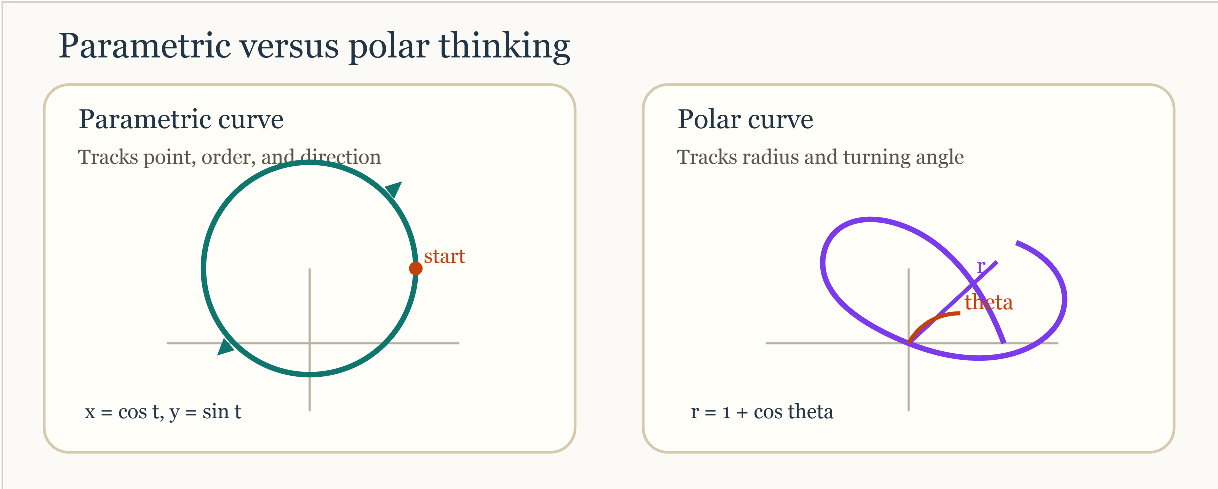
These equations trace the unit circle once counterclockwise. Eliminating the parameter gives

$$x^2 + y^2 = 1,$$

but the equation alone does not reveal:

- where the tracing starts,
- whether the motion is clockwise or counterclockwise,
- or whether the curve is traced once, twice, or only partially.

At $t = 0$, the point is `(1, 0)`. As `t` increases slightly, `y` becomes positive and `x` decreases, so the motion begins upward and leftward. That is the signature of counterclockwise orientation.



The same geometric set can have many parametrizations

The curve

$$x = \cos t, y = \sin t$$

and the curve

$$x = \cos(2t), y = \sin(2t)$$

describe the same circle, but not the same motion. The second parametrization traces the circle twice as fast with respect to `t`.

Likewise,

$$x = \cos(-t), y = \sin(-t)$$

traces the same circle in the opposite direction. This is why textbooks treat parametrization itself as mathematical data, not only as a disposable intermediate step.

Example: a parabola with built-in direction

Let

$$x = t, y = t^2 - 2t.$$

Eliminating the parameter yields

$$y = x^2 - 2x.$$

So the geometric set is an ordinary parabola, but the parametrization still matters: as `t` increases, the point moves from left to right with steadily increasing `x`. If the curve instead used $x = 1 - t$, the same parabola would be traced right to left.

When a parametrization is better than elimination

Eliminating the parameter is useful for recognizing a familiar shape, but it is often a poor first move. Keep the parametrization when you want to know:

- velocity and speed,
- tangent direction,
- repeated points or self-intersections,
- or how a model evolves over time.

One of the most common mistakes in this topic is to eliminate the parameter so quickly that the motion story is lost.

16.2 Calculus of parametric curves

Once a curve is parametrized, derivatives with respect to the parameter are immediate:

dx/dt and dy/dt.

The slope of the tangent line with respect to rectangular coordinates is then

$dy/dx = (dy/dt)/(dx/dt)$,

provided $dx/dt \neq 0$.

Why the slope formula is reasonable

If a tiny change dt produces changes dx and dy, then

$$dy/dx = (dy/dt)/(dx/dt)$$

is simply the chain rule for how `y` changes with `x` through the common parameter `t`. This formula preserves the central single-variable idea: slope is still change in vertical position divided by change in horizontal position.

Example: tangent line to a parametric curve

Suppose

$x = t^2 + 1, y = t^3 - t$.

Then

- $dx/dt = 2t$,
- $dy/dt = 3t^2 - 1$.

So

$dy/dx = (3t^2 - 1)/(2t)$.

At $t = 1$, the point on the curve is `(2, 0)` and the slope is

$(3 - 1)/2 = 1$.

Therefore the tangent line at $t = 1$ is

$y = x - 2$.

This example shows a typical workflow:

1. evaluate the point from $x(t)$ and $y(t)$,
2. compute the slope from dy/dt and dx/dt,
3. write the tangent line in point-slope form,
4. interpret the geometry.

Horizontal and vertical tangents

The curve has:

- a horizontal tangent when $dy/dt = 0$ and $dx/dt \neq 0$,
- a vertical tangent when $dx/dt = 0$ and $dy/dt \neq 0$.

These conditions are more subtle than the graph case because neither `x` nor `y` is primary. Both coordinates are evolving together.

Example: finding special tangent directions

Let

$x = t^2 - 1, y = t^3 - 3t$.

Then

- $dx/dt = 2t$,
- $dy/dt = 3t^2 - 3 = 3(t^2 - 1)$.

Horizontal tangents occur when $3(t^2 - 1) = 0$, so $t = 1$ or $t = -1$, provided $2t \neq 0$. Both values work.

Vertical tangents occur when $2t = 0$, so $t = 0$, provided $3(t^2 - 1) \neq 0$. Since $dy/dt = -3$ there, a vertical tangent does occur at $t = 0$.

The three special points are:

- $t = -1$: `(0, 2)`,
- $t = 0$: `(-1, 0)`,
- $t = 1$: `(0, -2)`.

Notice that the same `x`-value appears more than once. Parametric description handles that cleanly.

Second derivative for parametric curves

The second derivative with respect to `x` is not simply d^2y/dt^2. Because slope itself depends on `t`, we differentiate dy/dx with respect to `t` and divide again by dx/dt:

$d^2y/dx^2 = (d/dt(dy/dx))/(dx/dt)$.

This tells whether the curve bends upward or downward relative to rectangular axes. It is a bookkeeping-heavy formula, but the meaning is familiar: how fast is the slope changing as we move along the curve?

Example: concavity from a parametrization

Take

$x = t, y = t^3 - 3t$.

Then

$$dy/dx = dy/dt = 3t^2 - 3$$

because $dx/dt = 1$. Therefore

$d^2y/dx^2 = 6t$.

The curve is concave down for $t < 0$, concave up for $t > 0$, and changes concavity at $t = 0$. Parametric notation did not change the calculus ideas; it only changed the bookkeeping.

A diagnostic reading habit

When working with a parametrized curve, ask three questions in order:

1. Where is the point?
2. In which direction is it moving?

3. How is that direction changing?

Those questions correspond to:

- position,
- first derivative,
- second derivative or curvature.

That habit prevents the most common classroom error: mixing up the geometric curve with the motion along the curve.

16.3 Arc length, area, and models in parametric form

Parametric curves support the same accumulation ideas used elsewhere in calculus, but the local pieces now come from both coordinate functions.

Arc length of a parametric curve

If

$r(t) = \langle x(t), y(t) \rangle, a \leq t \leq b,$

then the arc length is

$L = \int_a^b sqrt((dx/dt)^2 + (dy/dt)^2)dt.$

This is the plane version of the speed integral from Chapter 12. The expression under the square root is the squared speed, so the formula says:

> total distance traveled = integral of speed.

Example: length of a semicubical path

Let

$x = t^2, y = t^3, 0 \leq t \leq 1.$

Then

- $dx/dt = 2t,$
- $dy/dt = 3t^2.$

So

$L = \int_0^1 \sqrt{4t^2 + 9t^4} dt = \int_0^1 t\sqrt{4 + 9t^2} dt.$

The point of this example is not the final antiderivative alone. It shows how geometry, derivatives, and integration collaborate:

- derivatives describe the local motion,
- the square root combines horizontal and vertical movement,
- the integral accumulates those local lengths.

Area under a parametric curve

If a parametric curve satisfies $x'(t) \geq 0$ on `[a, b]` and lies above the `x`-axis, then the area under the curve is

$A = \int_a^b y(t)x'(t) dt.$

This is just the rectangular-area idea

$$intydx$$

rewritten in parameter form. It reminds us that parametric calculus is usually ordinary calculus with an additional layer of description.

Example: parametric area

Suppose

$x = t^2, y = 1 + t, 0 \leq t \leq 2.$

Then $x'(t) = 2t$, so the area under the curve is

$A = \int_0^2 (1 + t)(2t) dt = \int_0^2 (2t + 2t^2) dt.$

The setup is more important than the arithmetic because the real structural step is recognizing that $dx = x'(t) dt$.

Projectile motion as a parametric model

Many physical motions are naturally parametric because time drives the system. For a projectile launched from `(0, 0)` with horizontal velocity $v_0 \cos \alpha$ and vertical velocity $v_0 \sin \alpha$, the standard model is

- $x(t) = v_0 \cos \alpha t,$
- $y(t) = v_0 \sin \alpha t - (1/2)gt^2.$

This parametrization encodes:

- horizontal linear motion,
- vertical uniformly accelerated motion,
- and the time structure of the path.

Eliminating `t` yields a parabola, but the parametrization retains the physical meaning of each term. That is why benchmark textbooks devote significant space to parametric motion before or alongside vector-valued functions.

Ellipses, orbits, and repeated geometry

A standard ellipse can be parametrized as

- $x = a\cos t$,
- $y = b\sin t$.

This is a good reminder that parametrization can capture symmetry directly. Many orbit models and periodic motions inherit this same structure even when the interpretation of `t` changes.

Local approximation still governs everything

The philosophy of the chapter remains the same as everywhere else in calculus:

- use derivatives to understand local behavior,
- use integrals to accumulate those local behaviors into global quantities.

Parametric notation does not introduce a new kind of mathematics so much as a new bookkeeping system for the same core ideas.

16.4 Polar coordinates and graphing

Rectangular coordinates describe a point by horizontal and vertical displacement. Polar coordinates describe a point by:

- distance from the origin `r`,
- angle from the positive `x`-axis θ.

So a polar point is written

(r, θ).

Converting between coordinate systems

The standard relationships are

- $x = r\cos\theta$,
- $y = r\sin\theta$,
- $r^2 = x^2 + y^2$,
- $\tan\theta = y/x$ when appropriate.

The first two equations are usually the most useful because they let us move back to rectangular coordinates whenever a curve is unfamiliar.

Multiple polar descriptions of the same point

Unlike rectangular coordinates, polar coordinates are not unique. The same point can be represented by:

- adding multiples of $2pi$ to the angle,
- or using a negative radius with angle shifted by π.

For example,

$(2, \pi/3), (2, 7pi/3)$, and $(-2, 4pi/3)$

all describe the same point.

This flexibility is useful, but it also creates errors. Students sometimes think a polar equation is wrong because the radius becomes negative. In fact, a negative radius simply points in the opposite angular direction.

Example: polar graph $r = 2$

The equation $r = 2$ means all points at distance `2` from the origin. That is a circle of radius `2` centered at the origin.

Example: polar graph $\theta = \pi/4$

This is the line through the origin making a `45` degree angle with the positive `x`-axis. Polar equations can describe lines and circles very efficiently when symmetry centers on the origin.

Symmetry tests that save time

For many textbook polar curves, the fastest route to a sketch is a symmetry test.

Common checks:

- replace θ by $-\theta$ to test symmetry about the `x`-axis,
- replace θ by $\pi - \theta$ to test symmetry about the `y`-axis,
- replace θ by $\theta + \pi$ or replace `r` by `-r` to test origin symmetry.

These tests are not magic rules. They are structured ways of asking whether the same set of points is produced after a geometric reflection or rotation.

Example: rose curve

Consider

$r = 2\cos(3theta).$

The oscillating cosine means the curve repeatedly expands and contracts as the angle turns. The factor `3` produces a three-petal structure for cosine over a full revolution. Benchmark textbooks devote many pages to such examples because polar graphing combines algebraic pattern recognition with geometric interpretation.

Example: cardioid

The equation

$$r = 1 + \cos\theta$$

produces a cardioid. The graph shows how a simple trigonometric adjustment in the radius can create a cusp and a strong directional bias.

Polar graphing becomes easier when you track:

- where $r = 0$,
- where `r` is maximal or minimal,
- sign changes in `r`,
- and symmetry.

16.5 Calculus in polar coordinates

Polar curves are not only graphing objects. They support tangent slopes, areas, and arc lengths.

Slope of a polar curve

Using

- $x = r\cos\theta$,
- $y = r\sin\theta$,

with $r = r(\theta)$, we differentiate with respect to θ:

- $dx/dtheta = r'\cos\theta - r\sin\theta$,

- $dy/dtheta = r' \sin\theta + r\cos\theta$.

Therefore

$$dy/dx = (r'\sin\theta + r\cos\theta)/(r'\cos\theta - r\sin\theta),$$

provided the denominator is nonzero.

This formula looks dense, but it comes from the same parametric idea as before. Polar curves are parametric curves in disguise, with θ playing the role of the parameter.

Example: slope on a cardioid

Let

$$r = 1 + \cos\theta.$$

Then $r' = -\sin\theta$. At $\theta = \pi/2$ we have:

- $r = 1$,
- $r' = -1$,
- $dx/dtheta = -1$,
- $dy/dtheta = 0$.

So the tangent there is horizontal.

The point is

$$(x, y) = (0, 1).$$

Even when the algebra is heavier than in rectangular coordinates, the meaning is unchanged: compute the point, compute the slope, interpret the tangent.

Area in polar coordinates

For a polar curve $r = f(\theta)$ with $\alpha \le \theta \le \beta$, the enclosed area is

$$A = (1/2)\int_a lpha^{\beta} r^2 dtheta.$$

This formula is one of the signature results of the topic. The factor $1/2$ appears because each thin sector behaves like half the product of two radii and the included angle when the angle slice is very small.

Example: area enclosed by a cardioid

For

$r = 1 + \cos\theta$,

the whole curve is traced for $0 \leq \theta \leq 2pi$, so

$A = (1/2)\int_0^{2pi}(1 + \cos\theta)^2 dtheta.$

Before integrating, pause and read the structure:

- the square comes from area scaling with length squared,
- the integral accumulates over angle,
- and the geometry is centered on the origin rather than on vertical slices.

This is exactly the kind of idea students remember when the formula is connected to geometry instead of treated as an isolated rule.

Why the factor `r` appears in local polar area

Chapter 14 discusses the `r` factor in double integrals from a multivariable viewpoint. In single-variable polar area, the same geometry is already present. A narrow sector with radius `r` and angle `dtheta` has arc length approximately `r dtheta`, so its area behaves like

$(1/2)r(rdtheta) = (1/2)r^2 dtheta.$

That is the local geometric reason behind both formulas. The single-variable area formula is not separate from the multivariable Jacobian idea; it is the same idea in an earlier form.

Arc length in polar coordinates

If $r = r(\theta)$, then the arc length from $\theta = \alpha$ to $\theta = \beta$ is

$L = \int_a lpha^\beta sqrt(r^2 + (dr/dtheta)^2)dtheta.$

This comes from the parametric arc-length formula after substituting

- $x = r\cos\theta$,
- $y = r\sin\theta$.

Example: arc length of a circle in polar form

For the circle $r = a$, we have $dr/dtheta = 0$, so

$L = \int_0^{2pi}\sqrt{a^2}dtheta = \int_0^{2pi} adtheta = 2pia.$

That check matters. A formula is more trustworthy when it reproduces a familiar special case cleanly.

Polar area versus polar arc length

Students often confuse the formulas because both involve `r` and integration over θ. Their units separate them:

- $r^2 dtheta$ has area units,
- $sqrt(r^2 + (dr/dtheta)^2)dtheta$ has length units.

Unit checking is especially useful in coordinate-change topics because the formulas are easy to memorize incorrectly.

16.6 Curvature and osculating circles

Slope measures tilt. Curvature measures turning.

A straight line can have nonzero slope but zero curvature. A curve with rapidly changing direction can have large curvature even when its slope is momentarily zero.

Curvature as rate of turning

At an intuitive level, curvature answers:

> how quickly is the tangent direction changing as we move along the curve?

High curvature means sharp turning. Low curvature means the path is locally close to a line.

Curvature of a plane curve

For a parametrized plane curve $r(t) = \langle x(t), y(t) \rangle$, one standard curvature formula is

$kappa = |x'y'' - y'x''|/((x')^2 + (y')^2)^{(}3/2)$.

The exact formula matters less than the interpretation:

- the numerator measures how strongly the velocity and acceleration fail to point in the same direction,
- the denominator normalizes by speed.

So curvature isolates turning from mere scaling of the parameter.

Example: curvature of a circle

For the circle

$x = a \cos t, y = a \sin t,$

the curvature is constant and equals `1/a`.

This is a beautiful sanity check:

- a smaller circle bends more sharply,
- a larger circle bends more gently,
- and a straight line corresponds to infinite radius and zero curvature.

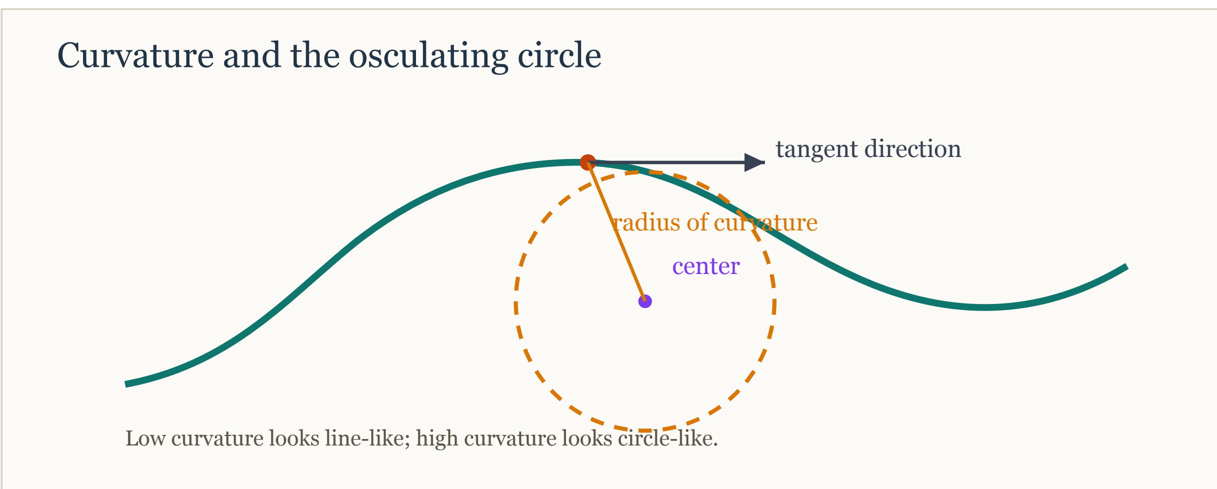
Osculating circles

At a point on a sufficiently smooth curve, the osculating circle is the circle that best matches the curve's local direction and turning. Its radius

$$\varrho = 1/\mathrm{kappa}$$

is called the radius of curvature.

This connects a complicated curve to a simple geometric model: near a point, the curve behaves most like a line when curvature is tiny and most like a small circle when curvature is large.



Why curvature belongs in a calculus text

Curvature is where several themes of the book meet:

- local linearity is not enough when turning matters,
- derivatives describe local behavior,
- and geometric quality can be measured quantitatively.

In physics, curvature influences turning forces. In engineering, curvature controls bending and design constraints. In computer graphics, curvature helps decide how fine a curve approximation must be.

Example: comparing two paths

Suppose one road is nearly straight and another bends sharply through a short interval. Even if both have similar slopes at one point, the second road has larger curvature. This is why drivers experience turning force based on how direction changes, not merely on the instantaneous slope of the road relative to the horizon.

Curvature and approximation

Linear approximation uses tangent lines. A better local geometric approximation sometimes uses:

- the tangent line for first-order behavior,
- the osculating circle for turning behavior.

This is another way to understand why some textbook chapters linger on curvature: it deepens the student's sense of what a derivative can and cannot tell us by itself.

Quick tactics

- Keep the parameter until you are sure the motion information is no longer needed.
- For tangent lines, evaluate the point and the slope separately.
- In polar graphing, find zeros of `r`, maxima or minima, and symmetry before plotting many points.
- Use units to distinguish area formulas from length formulas.
- Check special cases such as circles whenever a new formula looks complicated.

Chapter review

This chapter added three large ideas:

1. Parametric description keeps track of order, direction, and motion.
2. Polar coordinates reorganize the plane around angle and distance from the origin.
3. Curvature measures turning, not only tilting.

The chapter's main formulas are:

- $dy/dx = (dy/dt)/(dx/dt)$ for parametric curves,
- $L = intsqrt((x')^2 + (y')^2)dt$ for parametric arc length,
- $A = intyx'dt$ for area under a parametric curve when appropriate,
- $A = (1/2)intr^2dtheta$ for polar area,
- $L = intsqrt(r^2 + (r')^2)dtheta$ for polar arc length,

- and curvature formulas that quantify turning.

Mini projects

Project 1: compare three descriptions of the same curve

Choose a curve that can be described:

- explicitly,
- parametrically,
- and in polar form if possible.

Explain what each description makes easy to see and what each one hides.

Project 2: local geometry portfolio

Pick two curves from physics, biology, transportation, or design. For each, identify:

- tangent behavior,
- turning behavior,
- a useful local approximation,
- and one quantity that an engineer or scientist would actually care about.

Common traps

- Eliminating the parameter so early that orientation and repeated tracing disappear.
- Treating dy/dt as if it were the slope in the plane without dividing by dx/dt.
- Forgetting that polar coordinates are not unique.
- Mixing the polar area and arc-length formulas.
- Assuming a negative radius means an impossible point.
- Using curvature language when only slope has been computed.

Proof window: why $dy/dx = (dy/dt)/(dx/dt)$ is plausible

If `y` and `x` both depend on `t`, then small changes satisfy

- $dy \approx (dy/dt)dt$,
- $dx \approx (dx/dt)dt$.

Dividing gives

$$dy/dx \approx ((dy/dt)dt)/((dx/dt)dt) = (dy/dt)/(dx/dt),$$

provided dx/dt is not zero. This is the chain rule written as a local-rate comparison. The formula is therefore not a mysterious trick; it is the natural way to compare two rates driven by the same parameter.

Exercises

Warm-up: reading parametrizations and polar graphs

1. Describe the starting point and orientation of $x = \cos t, y = \sin t$ on $0 \le t \le 2pi$.
2. Explain how $x = \cos(2t), y = \sin(2t)$ differs from $x = \cos t, y = \sin t$ even though the geometric set is the same.
3. Convert the point $(2, 3pi/4)$ from polar to rectangular coordinates.
4. Give two different polar descriptions of the rectangular point `(-1, 0)`.
5. Sketch the curve $r = 2$.
6. Sketch the curve $\theta = \pi/3$.

Core skill: slopes, tangents, and local behavior

1. For $x = t^2 + 1, y = t^3 - t$, find the tangent line at $t = 1$.
2. For $x = t^2 - 1, y = t^3 - 3t$, find all points with horizontal tangents.
3. For the same curve, find all points with vertical tangents.
4. Compute d^2y/dx^2 for $x = t, y = t^3 - 3t$.
5. For $x = e^t, y = e^{-t}$, find dy/dx.
6. Explain geometrically why a parametrized curve can have the same point at two different parameter values.
7. For $x = 3\cos t, y = 2\sin t$, find the slope at $t = \pi/4$.
8. Determine where the ellipse in Exercise 13 has horizontal tangents.
9. Determine where it has vertical tangents.

Interpretation: arc length, area, and geometry

1. Set up the arc-length integral for $x = t^2, y = t^3, 0 \le t \le 1$.
2. Set up the area-under-the-curve integral for $x = t^2, y = 1 + t, 0 \le t \le 2$.
3. Explain why $inty(t)x'(t)dt$ reduces to $intydx$.
4. Describe a modeling setting in which a parametric description is clearly better than an explicit graph.
5. For projectile motion with $x(t) = 20t, y(t) = 30t - 16t^2$, explain what each term represents physically.

6. Explain why the parametrization of an orbit or vibration usually carries information that an eliminated rectangular equation does not.

Core skill: polar graphing and polar calculus

1. Convert $r = 3\cos\theta$ to a rectangular equation and identify the graph.
2. Find the points where $r = 1 + \cos\theta$ passes through the origin.
3. Test symmetry of $r = 2\sin\theta$.
4. Sketch a reasonable graph of $r = 1 + \cos\theta$.
5. Find the slope of $r = 1 + \cos\theta$ at $\theta = \pi/2$.
6. Set up the area integral for one full tracing of $r = 2\cos(3 theta)$.
7. Set up the area integral enclosed by $r = 1 + \cos\theta$.
8. Use the arc-length formula to write the length of $r = 2$ on $0 \le \theta \le 2pi$.
9. Explain in words why the factor r^2 belongs in polar area.

Challenge: curvature, comparison, and nonroutine reasoning

1. Show that the circle $x = a\cos t$, $y = a\sin t$ has constant curvature.
2. Compare the curvature of circles of radius `2` and `5`. Which bends more sharply and why?
3. Explain why curvature can be large at a point where the slope is zero.
4. Give an example of a curve with nonzero slope and zero curvature.
5. Describe why a tangent line is a first-order approximation while an osculating circle captures additional turning information.
6. A curve is traced twice with different speeds but the same geometric path. Explain which quantities change and which do not: tangent direction, speed, arc length of the traced path, curvature.

Modeling: motion, design, and coordinate choice

1. A camera drone must circle a target while gradually rising. Propose a parametric model and explain how each coordinate reflects the design goals.
2. A flower-shaped decorative path is modeled in polar coordinates. Explain why polar description is more natural than rectangular coordinates.
3. Write a short paragraph comparing a road-design concern that depends on slope with one that depends on curvature.
4. Build a one-page "which coordinate system should I use?" guide for explicit, parametric, and polar descriptions.

Reflection

Parametric and polar thinking broaden the student's picture of what a curve can be. A curve is not only a graph. It can be:

- a traced path,
- a coordinate transformation,
- a local-turning object,
- or a model of motion.

That expansion is one reason long mainstream calculus textbooks spend many pages here. The topic deepens both geometric intuition and derivative/integral fluency at the same time.

Chapter 17. Second-Order Differential Equations and Oscillation

Opening question

What happens when a model tells us how acceleration depends on position and velocity?

First-order differential equations describe many growth, decay, and flow processes. But many of the most recognizable physical systems are governed by second derivatives: springs, vibrating strings, electrical circuits, bending beams, and feedback controls. Comprehensive calculus texts therefore usually include a chapter on second-order differential equations and oscillation. This chapter adds that benchmark material while keeping the focus on modeling, interpretation, and classroom usefulness.

Learning goals

By the end of this chapter, you should be able to:

- interpret a second-order differential equation as an acceleration law,
- solve basic linear second-order equations with constant coefficients,
- recognize oscillatory solutions and connect them to sine and cosine,
- distinguish undamped, damped, and forced behavior conceptually,
- compare initial-value and boundary-value thinking,
- and explain how qualitative and numerical reasoning support exact formulas.

Preview questions

- Why do acceleration laws naturally create second derivatives?
- Why do exponentials and trigonometric functions appear so often in second-order models?
- What is the difference between "how a system starts" and "how a system is constrained at its endpoints"?
- How can a system oscillate forever in one model and settle down in another?

17.1 From acceleration laws to second-order equations

If $y(t)$ describes position, then:

- `y'(t)` is velocity,
- `y''(t)` is acceleration.

So any rule that links acceleration to position, velocity, or forcing produces a second-order differential equation.

Example: spring-mass law

A simple spring model often takes the form

$$my'' + ky = 0,$$

where:

- `m` is mass,
- `k` is spring stiffness,
- `y` is displacement from equilibrium.

The equation says acceleration is proportional to the restoring displacement and points toward equilibrium.

Example: damping

If a damping force proportional to velocity is included, the model becomes

$$my'' + cy' + ky = 0,$$

where `c` measures damping strength.

Now the system does not merely try to return to equilibrium. It also loses energy as it moves.

Why second-order models matter

Second-order equations appear whenever:

- restoring forces depend on displacement,
- inertia matters,
- or the system remembers not only where it is but how it is moving.

That makes them central in mechanics, engineering, and many scientific approximations.

Initial conditions for second-order equations

A first-order differential equation usually needs one initial condition. A second-order equation usually needs two because a system's future depends on:

- its initial position,
- and its initial velocity.

For example, a spring can start at the same location in two physically different states:

- released from rest,
- or shot through equilibrium at high speed.

Those are different motions because the initial velocity differs.

Example: interpreting data as conditions

If a mass begins `2` centimeters above equilibrium and is released from rest, the mathematical conditions are

- $y(0) = 2$,
- $y'(0) = 0$.

This translation from words to initial conditions is one of the most important modeling habits in the chapter.

17.2 Linear second-order equations with constant coefficients

The standard homogeneous model is

$$ay'' + by' + cy = 0,$$

where `a`, `b`, and `c` are constants and $a \neq 0$.

The main solution method uses the characteristic equation

$$ar^2 + br + c = 0.$$

Why exponentials appear

If we try a solution of the form $y = e^{rt}$, then:

- $y' = re^{rt}$,
- $y'' = r^2 e^{rt}$.

Substituting into the differential equation gives

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0.$$

Since e^{rt} never vanishes, we need

$$ar^2 + br + c = 0.$$

That is the origin of the characteristic equation. The method works because exponentials reproduce themselves under differentiation.

Case 1: two distinct real roots

If the characteristic equation has distinct real roots r_1 and r_2, then

$$y = C_1e^{r_1t} + C_2e^{r_2t}.$$

Example: distinct real roots

Solve

$$y'' - 5y' + 6y = 0.$$

The characteristic equation is

$$r^2 - 5r + 6 = 0,$$

so the roots are `2` and `3`. Therefore

$$y = C_1e^{2t} + C_2e^{3t}.$$

The exact constants depend on the initial conditions.

Case 2: repeated real root

If the characteristic equation has a repeated root `r`, then the solution is

$$y = (C_1 + C_2t)e^{rt}.$$

The factor `t` supplies the second independent behavior needed for a two-parameter solution family.

Example: repeated root

Solve

$$y'' - 4y' + 4y = 0.$$

The characteristic equation is

$$r^2 - 4r + 4 = (r - 2)^2,$$

so the repeated root is `2`. Hence

$$y = (C_1 + C_2 t)e^{2t}.$$

Case 3: complex roots

If the characteristic roots are

$$r = \alpha + -\beta i,$$

then the real-valued solution is

$$y = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t).$$

This is where oscillation enters. Sine and cosine are the natural signatures of complex-conjugate roots.

Example: oscillatory solution

Solve

$$y'' + 9y = 0.$$

The characteristic equation is

$$r^2 + 9 = 0,$$

so $r = + - 3i$. Therefore

$$y = C_1 \cos 3t + C_2 \sin 3t.$$

The motion oscillates with angular frequency `3`.

Strategy chart

For constant-coefficient linear equations, the workflow is:

1. write the characteristic equation,
2. classify its roots,
3. write the corresponding solution family,
4. use initial conditions to determine constants,
5. interpret the physical behavior from the root type.

This last step is what turns the chapter from algebra to modeling.

17.3 Harmonic motion and energy

The equation

$$y'' + \omega^2 y = 0$$

models ideal undamped simple harmonic motion.

General solution

The solution is

$y = C_1 \cos \omega t + C_2 \sin \omega t.$

This can also be rewritten as

$y = A \cos(\omega t - \phi),$

where:

- `A` is amplitude,
- ω is angular frequency,
- $2pi/\omega$ is period,
- ϕ is phase shift.

Why oscillation appears

When the restoring force points toward equilibrium and is proportional to displacement, the system overshoots equilibrium rather than stopping there. Inertia carries the system through the center, then the restoring force reverses direction and pulls it back. The result is repeated motion.

Example: solving from initial data

Solve

$y'' + 4y = 0, y(0) = 3, y'(0) = -2.$

The solution family is

$y = C_1 \cos 2t + C_2 \sin 2t.$

From $y(0) = 3$, we get $C_1 = 3$.

Differentiate:

$y' = -2C_1 \sin 2t + 2C_2 \cos 2t.$

Then $y'(0) = 2C_2 = -2$, so $C_2 = -1$.

Therefore

$y = 3\cos 2t - \sin 2t.$

Energy viewpoint

For the ideal spring equation $my'' + ky = 0$, the total mechanical energy is

$E = (1/2)m(y')^2 + (1/2)ky^2.$

The first term is kinetic energy and the second is potential energy. In the ideal undamped model, energy moves back and forth between these forms while the total remains constant.

This is pedagogically valuable because it teaches students to read a differential equation as a law of exchange between quantities, not only as a formula to solve.

Period, frequency, and units

Oscillation formulas are easier to remember when units are checked.

- ω has units of radians per unit time,
- $2pi/\omega$ has time units,
- amplitude has the same units as displacement.

Unit awareness catches many mistakes in spring and vibration problems.

17.4 Damping, forcing, and resonance

Real systems usually lose energy and may also be driven by external forces.

Damped motion

The damped model

$$my + cy + ky = 0$$

has three qualitative regimes:

- overdamped,
- critically damped,

- underdamped.

These regimes are determined by the discriminant of the characteristic equation.

Overdamped behavior

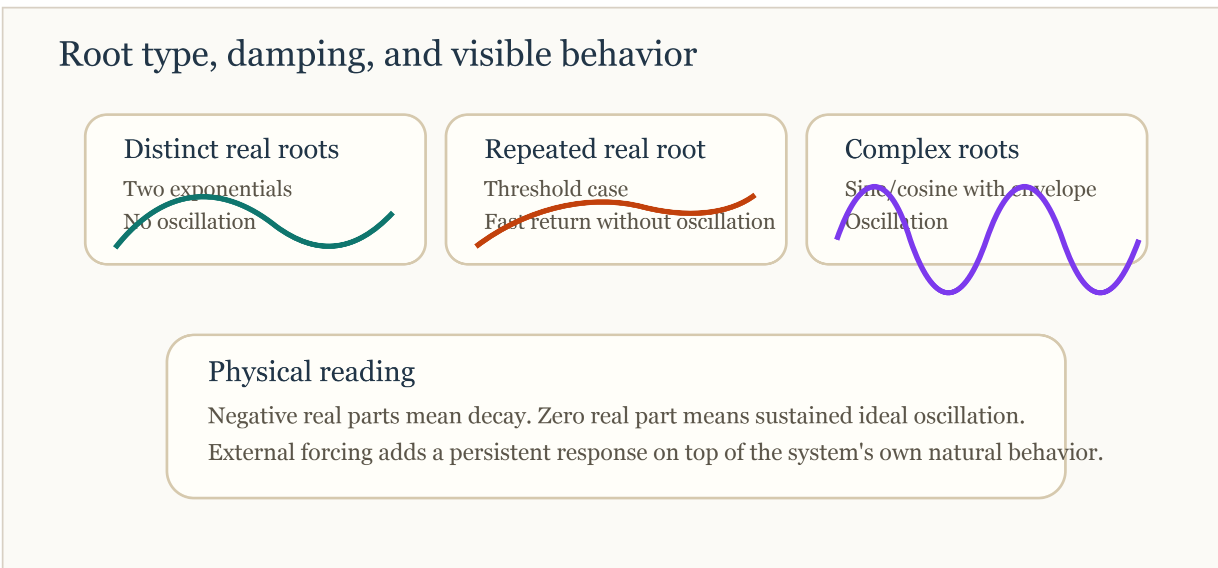
If damping is strong, the system returns to equilibrium without oscillating. The motion is sluggish but monotone after a point.

Critically damped behavior

At the threshold between oscillation and nonoscillation, the system returns to equilibrium as fast as possible without overshooting in the model. This case matters in design because it balances speed and stability.

Underdamped behavior

If damping is present but not too strong, the system oscillates while its amplitude gradually decays. This is often the most realistic classroom model for a vibrating object in air or fluid.



Example: underdamped equation

Solve qualitatively

$y'' + 2y' + 10y = 0.$

The characteristic equation is

$r^2 + 2r + 10 = 0,$

with roots

$r = -1 + -3i.$

So the solution has the form

$y = e^{-t}(C_1 \cos 3t + C_2 \sin 3t).$

The exponential factor shrinks the oscillation envelope over time. That is the visual signature of underdamping.

Example: critically damped return

Solve

$y'' + 4y' + 4y = 0, y(0) = 1, y'(0) = 0.$

The characteristic equation is

$r^2 + 4r + 4 = (r + 2)^2,$

so the repeated root is `−2`. The solution family is

$y = (C_1 + C_2 t)e^{-2t}.$

From $y(0) = 1$, we get $C_1 = 1$.

Differentiate:

$y' = (C_2 - 2C_1 - 2C_2 t)e^{-2t}.$

Using $y'(0) = 0$ gives

$C_2 - 2 = 0,$

so $C_2 = 2$. Therefore

$y = (1 + 2t)e^{-2t}.$

The key behavior is qualitative: the solution returns to equilibrium without oscillating, but faster than a sluggish overdamped response.

Forced motion

If an external input drives the system, we model

$my'' + cy' + ky = F(t).$

Now the solution contains two pieces:

- a transient part determined by the homogeneous equation,

- a forced response determined by the driving term.

In physical language:

- the system's own natural behavior tries to fade out,
- the persistent input keeps pushing.

Example: a nonresonant forced response

Solve

$y'' + y = 2\cos 2t.$

The homogeneous equation $y'' + y = 0$ has solution

$y_h = C_1 \cos t + C_2 \sin t.$

For a particular solution, try $y_p = A\cos 2t + B\sin 2t$. Then

$y_p'' = -4A\cos 2t - 4B\sin 2t,$

so

$y_p'' + y_p = -3A\cos 2t - 3B\sin 2t.$

Matching $2\cos 2t$ gives

- $-3A = 2,$
- $-3B = 0.$

Thus $A = -2/3$ and $B = 0$, so

$y = C_1 \cos t + C_2 \sin t - (2/3)\cos 2t.$

This example is valuable because it shows how the natural response and the forced response coexist in one formula.

Resonance

When the driving frequency is close to the system's natural frequency, oscillations can become large. This effect is called resonance.

In an undamped ideal model, resonance can produce unbounded growth in amplitude. In real systems, damping and nonlinear effects limit the growth, but the design lesson remains: matching frequencies can have dramatic consequences.

Why this belongs in a calculus chapter

Resonance is a strong example of why differential equations matter. The formulas are not decorative. They explain why bridges sway, why headphones vibrate, why suspension systems are tuned carefully, and why oscillatory devices must be designed around their natural frequencies.

17.5 Boundary-value problems and mode ideas

So far the chapter has emphasized initial-value problems: specify the state at one starting time and predict the future.

A boundary-value problem instead specifies conditions at different points, often at the ends of an interval.

Example: endpoint constraints

Consider

$$y'' + \lambda y = 0, y(0) = 0, y(L) = 0.$$

This is not asking how a motion begins. It is asking which shapes or modes satisfy endpoint constraints.

Why boundary-value problems matter

Boundary-value problems appear in:

- vibrating strings with fixed endpoints,
- steady-state temperature problems,
- beam deflection,
- wave modes,
- and many PDE models built from ODE pieces.

This is one of the bridges from elementary differential equations to more advanced applied mathematics.

A simple mode calculation

For

$$y'' + \lambda y = 0, y(0) = 0, y(\pi) = 0,$$

the only nontrivial solutions occur for special values of λ. If $\lambda = n^2$ for positive integers `n`, then

$$y = C\sin(nt)$$

satisfies both endpoint conditions.

The important idea is not memorizing the details. It is realizing that:

- endpoint constraints select special frequencies,
- and those special modes organize the behavior of larger systems.

Initial-value versus boundary-value thinking

The contrast is worth making explicit:

- initial-value problems ask for evolution from a starting state,
- boundary-value problems ask for shapes or states that satisfy distributed constraints.

Comprehensive calculus texts usually preview this distinction because it prepares students for differential equations, PDEs, physics, and numerical analysis.

17.6 Qualitative and numerical perspectives

Not every second-order equation yields a neat closed form, and even when one exists, the formula may not be the most informative first tool.

From second order to first-order systems

If we let $v = y'$, then the equation

$$y \quad = f(t, y, y \;)$$

can be rewritten as the first-order system

- $y' = v$,
- $v' = f(t, y, v)$.

This viewpoint turns position and velocity into a combined state. It is the natural language for numerical simulation and phase-plane reasoning.

Phase-plane intuition

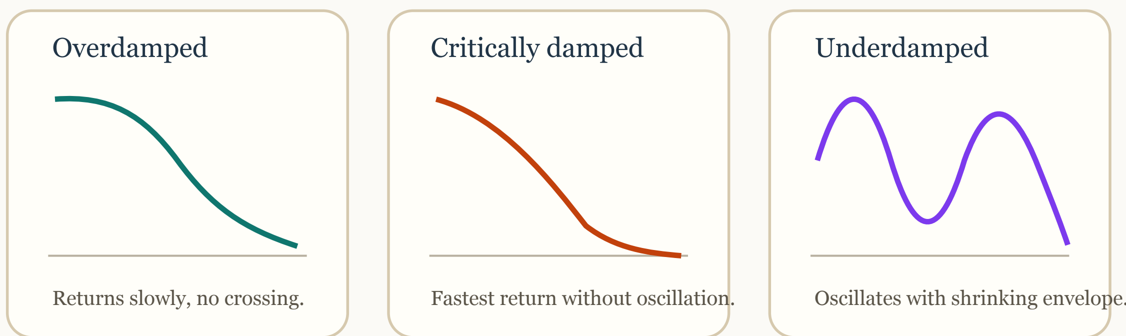
In a position-velocity plane:

- closed loops suggest sustained oscillation,

- inward spirals suggest damped oscillation,
- and direct return paths suggest strong damping without oscillation.

This is one reason long textbooks often invest many pages here. Students learn that a differential equation can be understood geometrically, not only symbolically.

Three damping regimes



Numerical stepping

A second-order equation can be stepped numerically by tracking both position and velocity. At a conceptual level:

1. compute acceleration from the current state,
2. update velocity over a small time step,
3. update position,
4. repeat.

This is the second-order analogue of Euler's Method from Chapter 11.

Example: one numerical step from a spring state

Suppose

$y'' = -4y - 0.5y'$,

with initial state

- $y(0) = 1$,
- $y'(0) = 0$.

Let $v = y'$. Then the first-order system is

- $y' = v$,
- $v' = -4y - 0.5v$.

At the initial state, the acceleration is

$v'(0) = -4(1) - 0.5(0) = -4.$

With a step size $h = 0.1$, a simple Euler update gives

- $v(0.1) \approx 0 + 0.1(-4) = -0.4,$
- $y(0.1) \approx 1 + 0.1(0) = 1.$

Even this crude first step says something useful: the system begins by accelerating downward from positive displacement, exactly as the restoring-force model predicts.

Model trust and model limits

Second-order models can be remarkably effective, but they depend on assumptions:

- linear restoring forces,
- constant coefficients,
- small oscillations,
- negligible nonlinear effects,
- and accurately measured parameters.

A powerful calculus habit is to separate three questions:

1. Did we solve the equation correctly?
2. Did we choose the right equation?
3. Over what range should we trust that choice?

That distinction matters as much as any algebra in the chapter.

Quick tactics

- Translate words into initial or boundary conditions before solving.
- Let the root type of the characteristic equation tell the behavior story.
- For oscillation problems, compute period and damping interpretation, not only constants.
- In forced problems, separate transient response from persistent driving response.
- Use units and physical meaning to check whether an answer is plausible.

Chapter review

This chapter added the following benchmark topics:

1. second-order acceleration models,
2. characteristic equations for constant-coefficient ODEs,
3. harmonic motion and energy exchange,
4. damping and forcing,
5. resonance,
6. and the contrast between initial-value and boundary-value problems.

The central interpretive lesson is that root types and coefficients encode behavior:

- real roots suggest nonoscillatory exponential behavior,
- repeated roots suggest threshold behavior,
- complex roots suggest oscillation,
- negative real parts suggest decay,
- forcing terms create sustained response.

Mini projects

Project 1: design a suspension story

Choose a physical suspension, spring, or vibration system. Estimate reasonable values or symbolic roles for mass, damping, and stiffness, then explain which qualitative regime the design should target and why.

Project 2: resonance case study

Write a two-page note describing a real system in which resonance is useful or dangerous. Your note should identify:

- the natural frequency,
- the driving mechanism,
- the role of damping,
- and how calculus helps predict the outcome.

Common traps

- Treating second-order initial conditions as if one condition were enough.
- Solving the characteristic equation correctly but failing to interpret the root type.
- Forgetting the factor `t` in the repeated-root case.
- Confusing angular frequency with period.
- Treating any oscillatory formula as undamped even when an exponential envelope is present.

- Ignoring units and physical meaning in model parameters.

Proof window: why repeated roots need the factor `t`

When the characteristic equation has a repeated root `r`, the solution e^{rt} is not enough by itself because a second-order equation needs a two-parameter family. The extra factor te^{rt} provides a second independent function. One quick plausibility check is substitution: differentiating te^{rt} introduces both e^{rt} and te^{rt} terms, creating the additional flexibility needed to satisfy two independent conditions.

Exercises

Warm-up: reading and interpreting models

1. Explain why $my'' + ky = 0$ is an acceleration law.
2. State the physical meaning of the coefficients `m`, `c`, and `k` in $my'' + cy' + ky = 0$.
3. Give a real setting where two initial conditions are necessary.
4. Explain the difference between initial conditions and boundary conditions.
5. Describe in words what resonance means.
6. Describe in words what damping means.

Core skill: characteristic equations and exact solutions

1. Solve $y'' - 5y' + 6y = 0$.
2. Solve $y'' - 4y' + 4y = 0$.
3. Solve $y'' + 9y = 0$.
4. Solve $y'' + 4y = 0$ with $y(0) = 3, y'(0) = -2$.
5. Solve $y'' + 2y' + 10y = 0$.
6. Solve $y'' - 2y' + 5y = 0$.
7. For $y'' + 16y = 0$, find the period of oscillation.
8. For $y'' + 25y = 0$, write the solution satisfying $y(0) = 0, y'(0) = 10$.
9. For $y'' - y' - 6y = 0$, classify the root type before writing the solution.

Interpretation: oscillation, decay, and behavior

1. Explain why complex roots lead to oscillation.
2. Explain why an exponential factor e^{-t} causes amplitude decay.
3. Compare qualitatively the solutions of $y'' + 9y = 0$ and $y'' + 2y' + 9y = 0$.
4. Describe what "critically damped" means in a design context.

5. Explain why the solution to $y'' + \omega^2 y = 0$ can be written using either sine-and-cosine form or amplitude-phase form.
6. A system oscillates with period π. What is its angular frequency?
7. A spring system returns slowly to equilibrium without crossing it. Which damping regime is most plausible?

Challenge: boundary values, modes, and reasoning

1. Show that nontrivial solutions of $y'' + \lambda y = 0, y(0) = 0, y(\pi) = 0$ require special values of λ.
2. Explain why the repeated-root case sits at a threshold between two qualitatively different behaviors.
3. Compare the roles of the discriminant in a quadratic equation and in a second-order constant-coefficient ODE.
4. Give an example of a second-order equation whose solution should oscillate and an example whose solution should not. Justify each without solving fully.
5. Explain why root type is often more informative than the exact constants when sketching long-term behavior.
6. Write a short note describing how a phase-plane picture distinguishes undamped from damped motion.

Modeling: forcing, resonance, and computation

1. A child pumps a swing periodically. Explain how forcing and natural frequency interact.
2. Write a second-order model for a mass on a spring with damping and external periodic force. Label each term in words.
3. Explain why a bridge or tall structure designer cares about resonance even if the exact textbook model is simplified.
4. Convert $y'' = -4y - 0.5y'$ into a first-order system using $v = y'$.
5. Describe how a numerical step for a second-order equation updates both position and velocity.
6. Create a one-page checklist for deciding whether a linear second-order model is reasonable for a given physical situation.

Reflection

Second-order differential equations enlarge the student's picture of calculus in two directions at once. They introduce a richer class of equations and they show more clearly that derivatives are laws of behavior, not only tools for graphing. Oscillation, damping, resonance, and mode selection are among the most recognizable outputs of applied calculus, which is why benchmark texts devote so much space to them.

Chapter 18. Improper Integrals and Long-Tail Behavior

Opening question

Can an infinite region still have a finite accumulated total?

Calculus students eventually face two situations that strain the ordinary definite-integral picture:

- the interval of integration is unbounded,
- or the integrand becomes unbounded inside the interval.

In both cases the integral is called improper. Mainstream calculus books usually devote a full chapter to this topic because it links area, convergence, comparison, infinite series, probability, and model interpretation. This chapter adds that missing benchmark material.

Learning goals

By the end of this chapter, you should be able to:

- define an improper integral as a limit of proper integrals,
- decide whether basic improper integrals converge or diverge,
- use comparison and `p`-integral logic effectively,
- connect improper integrals to the integral test for series,
- estimate tails and remainder sizes,
- and interpret improper integrals in probability and modeling settings.

Preview questions

- Why does "infinite width" not automatically force infinite area?
- Why can some singular integrands still produce finite accumulation?
- How does comparison with a simpler benchmark integrand control convergence?
- What changes when an improper integral is used to model a long tail rather than a bounded region?

18.1 Improper integrals on unbounded intervals

A proper definite integral

$$\int_a^b f(x)dx$$

uses a finite interval and a well-behaved integrand. If the upper limit is infinite, the integral is defined through a limit:

$\int_a^i nfinityf(x)dx = \lim_(b \to infinity) \int_a^b f(x)dx,$

provided the limit exists.

Why this is the right definition

The integral from `a` to `b` measures accumulation up to a finite cutoff. Letting `b` grow asks whether the accumulated total settles to a finite value. That is the core convergence question.

Example: a convergent improper integral

Consider

$\int_1^i nfinity1/x^2dx.$

For finite `b`,

$\int_1^b x^(- 2)dx = [-1/x]_1^b = 1 - 1/b.$

Now let $b \to infinity$. Since $1/b \to 0$, the integral converges to `1`.

This is one of the most important examples in the chapter because it proves something initially counterintuitive:

> infinite horizontal extent does not force infinite area.

Example: a divergent improper integral

Now consider

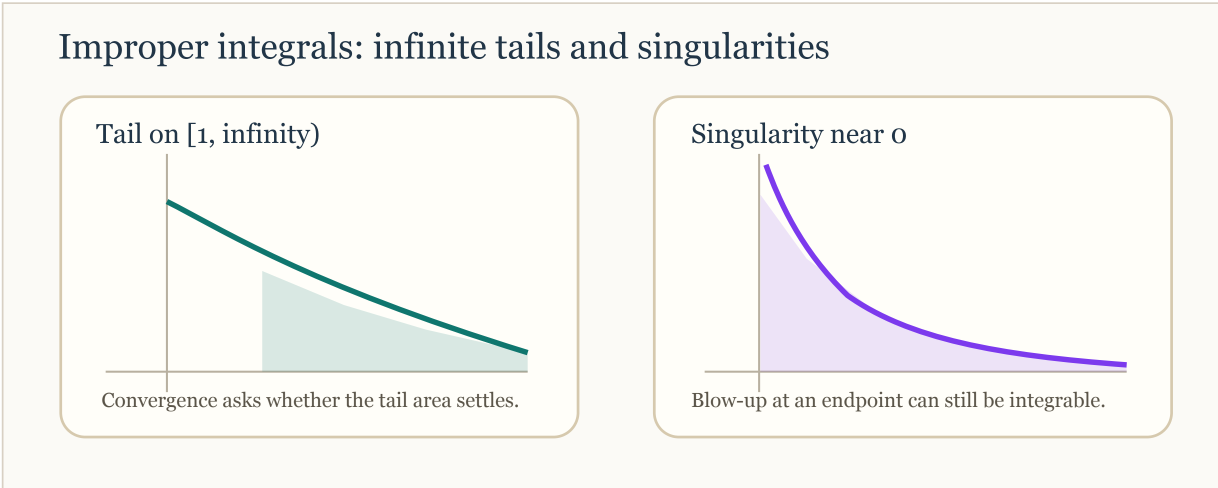
$\int_1^i nfinity1/xdx.$

For finite `b`,

$\int_1^b 1/xdx = \ln b.$

As $b \to \infty$, $\ln b \to \infty$, so the integral diverges.

The difference between `1/x` and $1/x^2$ is therefore qualitative, not merely computational. One tail is too thick to accumulate finitely; the other is thin enough.



Partial accumulation and tail interpretation

When the improper integral converges, the graph still stretches forever. The interpretation is not that the tail disappears. It is that the additional contribution from farther and farther regions becomes arbitrarily small.

This "small tail" language is often more intuitive for students than the raw limit statement.

Two-sided infinite intervals

If an interval extends in both directions, such as

$$\int_{-\infty}^{\infty} f(x)\,dx,$$

the correct approach is to split the integral:

$$\int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx.$$

Both pieces must converge for the total improper integral to converge. This is not just technical bookkeeping. Without splitting, cancellation can hide divergence.

18.2 Singular integrands and endpoint blow-up

An integral is also improper when the integrand becomes unbounded at an endpoint or interior point.

For example,

$$\int_0^1 1/\sqrt{x}\,dx$$

is improper because $1/\sqrt{x}$ blows up as $x \to 0^+$.

The definition is again a limit:

$\int_0^1 1/\sqrt{x}\,dx = \lim_{(a \to 0^+)} \int_a^1 x^{(-1/2)}dx,$

if the limit exists.

Example: convergent singularity

Compute

$\int_0^1 1/\sqrt{x}\,dx.$

For $a > 0$,

$\int_a^1 x^{(-1/2)}dx = [2\sqrt{x}]_a^1 = 2 - 2\sqrt{a}.$

As $a \to 0^+$, the value approaches `2`. So the integral converges.

This surprises many students because the function becomes infinite at $x = 0$. But the blow-up happens on such a narrow part of the interval that the total accumulated area remains finite.

Example: divergent singularity

Now consider

$\int_0^1 1/x\,dx.$

For $a > 0$,

$\int_a^1 1/x\,dx = -\ln a.$

As $a \to 0^+$, this grows without bound, so the integral diverges.

Again, the qualitative distinction is the main point:

- some singularities are integrable,
- some are not.

Interior singularities

If the integrand blows up inside the interval, split at the singular point. For example,

$$\int_0^2 1/\sqrt{|x-1|}dx$$

must be written as

$\int_0^1 1/\sqrt{1-x}dx + \int_1^2 1/\sqrt{x-1}dx.$

Each side is checked separately.

Why splitting matters

Improper integrals are convergence problems, so hidden cancellation is dangerous. Splitting an integral at every infinite endpoint or singular point is the correct structural habit.

18.3 `p`-integrals and comparison reasoning

The simplest convergence benchmarks come from the family

$$\int_1^i nfinity1/x^p dx$$

and

$\int_0^1 1/x^p dx.$

The `p`-integral rules

For integrals on `[1, infinity)`:

- convergence occurs when $p > 1$,
- divergence occurs when $p \leq 1$.

For integrals on `(0, 1]`:

- convergence occurs when $p < 1$,
- divergence occurs when $p \geq 1$.

These rules are not arbitrary. They are the benchmark against which many rational, radical, and exponential tails can be compared.

Why the threshold changes

Near infinity, larger powers of `x` make the integrand decay faster, which helps convergence.

Near `0`, larger powers of `x` in the denominator make the singularity stronger, which hurts convergence.

This is the kind of directional reasoning that keeps the chapter from becoming a memorization exercise.

Direct comparison test

If $0 \le f(x) \le g(x)$ for large `x` and

$$\int_a^{i} nfinity g(x)dx$$

converges, then

$$\int_a^{i} nfinity f(x)dx$$

also converges.

Likewise, if $f(x) \ge g(x) \ge 0$ and the smaller comparison integral diverges, then the larger one also diverges.

Example: comparison at infinity

Determine whether

$$\int_1^{i} nfinity 1/(x^2 + 1)dx$$

converges.

Since

$$0 < 1/(x^2 + 1) \le 1/x^2$$

for $x \ge 1$, and $\int_1^{i} nfinity 1/x^2 dx$ converges, the given integral converges by comparison.

Example: comparison for divergence

Determine whether

$$\int_1^{i} nfinity 1/(x + 3)dx$$

converges.

For $x \geq 1$,

$1/(x+3) \geq 1/(4x)$.

Because $\int_1^\infty 1/x\,dx$ diverges, so does $\int_1^\infty 1/(4x)\,dx$, and therefore the given integral diverges by comparison.

Limit comparison

When two functions behave similarly for large `x`, limit comparison is often cleaner. If

$$\lim_{x \to \infty} f(x)/g(x) = L$$

with $0 < L < \infty$, then `f` and `g` share the same convergence behavior.

This is especially useful for rational functions, where highest-degree terms dominate.

Example: rational tail

Classify

$\int_1^\infty (3x^2+1)/(x^5-2x+7)\,dx$.

The highest-degree behavior is like $3/x^3$, so comparison with $1/x^3$ is natural. Since $p = 3 > 1$, the integral converges.

The point is not to guess recklessly. The point is to identify the dominant tail behavior and compare it with a benchmark whose convergence is already known.

Example: limit comparison in one line of thought

Classify

$\int_1^\infty (2x+5)/(x^2+1)\,dx$.

Compare with `1/x`. Since

$\lim_{x \to \infty} [(2x+5)/(x^2+1)]/(1/x) = \lim_{x \to \infty} (2x^2+5x)/(x^2+1) = 2$
,

the integrand has the same tail behavior as `1/x`. Because

$$\int_1^\infty 1/x\,dx$$

diverges, the given integral also diverges.

This is a useful example because the limit comparison test can replace a messier direct antiderivative calculation.

18.4 Integral test and tail estimates

Improper integrals connect directly to infinite series through the integral test.

Integral test

If `f` is continuous, positive, and decreasing on `[1, infinity)` and $a_n = f(n)$, then

$$suma_n$$

and

$$\int_1^i nfinityf(x)dx$$

either both converge or both diverge.

This is powerful because it allows a discrete question to be answered by a continuous benchmark.

Example: $sum 1/n^2$

Using $f(x) = 1/x^2$, the integral

$$\int_1^i nfinity1/x^2dx$$

converges, so $sum 1/n^2$ converges by the integral test.

Example: $sum 1/n$

Using $f(x) = 1/x$, the improper integral diverges, so the harmonic series diverges.

This chapter therefore explains a convergence fact that students encountered earlier in Chapter 10, but now with a continuous-accumulation lens.

Tail estimates

For a convergent positive decreasing series, the remainder after `N` terms satisfies

$\int_(N + 1)^i nfinityf(x)dx \le R_N \le \int_N^i nfinityf(x)dx.$

So improper integrals do not only classify convergence. They also measure how much total mass remains in the tail.

Example: tail bound for $sum 1/n^2$

After `N` terms,

$R_N \leq \int_N^i nfinity 1/x^2 dx = 1/N.$

So if we want the remainder below `0.01`, it is enough to take $N \geq 100$.

This is a practical textbook point. Convergence is not only yes-or-no. Often we want to know how much error remains after truncation.

Why the hypotheses matter

The positivity and monotonicity conditions in the integral test are not decoration. They ensure the area picture actually mirrors the staircase of partial sums in a controlled way.

18.5 Probability tails, decay laws, and the gamma function

Improper integrals are not merely chapter-end curiosities. They appear whenever total accumulation occurs over an infinite domain.

Probability density on an infinite interval

A probability density $p(x)$ on `[0, infinity)` must satisfy

- $p(x) \geq 0$,
- $\int_0^i nfinity p(x) dx = 1$.

The integral is usually improper because the domain is unbounded.

Example: exponential density

The function

$p(x) = \lambda e^(-\lambda x)$ for $x \geq 0$

is a probability density because

$\int_0^i nfinity \lambda e^(-\lambda x) dx = 1.$

This example matters because the same exponential form appears in waiting-time models, decay laws, and reliability theory.

Example: a tail probability

If

$p(x) = e^{-x}$ for $x \geq 0$,

then the probability that $X > 3$ is

$P(X > 3) = \int_3^i nfinity e^{-x} dx = [-e^{-x}]_3^i nfinity = e^{-3}$.

This is one of the cleanest bridges from improper integrals to risk language. The graph extends forever, but the tail beyond `3` has a finite and exactly measurable weight.

Long-tail interpretation

Even though the exponential density extends forever, most of the probability lies near the origin when λ is not too small. Improper integration makes that statement exact by turning a long tail into a measurable quantity.

Mean value on an infinite domain

Expected value is another improper integral:

$$E[X] = \int_0^i nfinity \times p(x)dx$$

when that integral converges.

This is one reason long calculus texts emphasize improper integrals. They are the doorway from geometric area language to probability, statistics, and applied modeling.

Gamma-function preview

One famous improper integral is

$Gamma(s) = \int_0^i nfinity x^(s - 1)e^(- x)dx$ for $s > 0$.

This function extends factorial-like behavior to noninteger inputs. For positive integers `n`,

$Gamma(n) = (n - 1)!$.

Example: checking one gamma value

At $s = 1$,

$$Gamma(1) = \int_0^i nfinitye^{-x}dx = 1.$$

That matches the factorial rule because $(1 - 1)! = 0! = 1$.

The example is small, but it helps demystify the gamma function by showing that it is not an abstract miracle. It is an improper integral that really does reproduce familiar factorial behavior.

The calculation belongs more naturally to a later course, but the example is worth including because it shows the conceptual reach of improper integrals: from area language to special functions and advanced modeling.

Why applications belong here

A benchmark calculus chapter is stronger when it does not stop at convergence classification. Students should also see what finite versus infinite total accumulation means in context:

- total probability,
- expected waiting time,
- finite energy,
- or long-tail risk.

Quick tactics

- Rewrite every improper integral as a limit before computing.
- Split at every infinite endpoint or singular point.
- Compare with $1/x^p$ whenever the tail looks rational or power-like.
- Near infinity, ask how fast the function decays; near `0`, ask how violently it blows up.
- Use integral-test remainders when a convergent series needs an error estimate.

Chapter review

This chapter added four big ideas:

1. improper integrals are limits of proper integrals,
2. convergence depends on tail or singular behavior, not only on interval size,
3. comparison with benchmark integrands is the main strategic tool,
4. improper integrals link continuous accumulation to infinite series and probability models.

The most important benchmark examples are:

- $\int_1^{i} nfinity 1/x^2 dx$ converges,
- $\int_1^{i} nfinity 1/x dx$ diverges,
- $\int_0^1 1/\sqrt{x} dx$ converges,
- $\int_0^1 1/x dx$ diverges.

Mini projects

Project 1: long-tail storytelling

Choose two positive functions on `[1, infinity)` that both approach `0`, but whose improper integrals behave differently. Explain the difference in plain language and with a comparison argument.

Project 2: probability beyond the picture

Write a short note showing how improper integrals move the interpretation of area from geometry to probability. Your note should include:

- a density on an infinite domain,
- a tail probability,
- and one explanation of why finite total probability does not conflict with infinite support.

Common traps

- Treating an improper integral like an ordinary definite integral without limits.
- Forgetting to split at singular points.
- Using comparison without checking positivity assumptions.
- Applying the `p`-integral threshold for infinity to a near-zero singularity without reversing the logic.
- Declaring convergence from a finite-looking antiderivative before taking the limit.

Proof window: why $\int_1^{i} nfinity 1/x dx$ diverges

For any finite `b`,

$\int_1^b 1/x dx = \ln b.$

As `b` grows, $\ln b$ grows without bound, though slowly. This proof matters because it defeats a common intuition that "anything shrinking to zero should have finite total area." Shrinking to zero is not enough; the decay must be fast enough.

Exercises

Warm-up: reading improper structure

1. Explain why $\int_1^i nfinity1/x^2dx$ is improper.
2. Explain why $\int_0^1 1/\sqrt{x}dx$ is improper.
3. Describe the difference between an infinite interval and an infinite integrand.
4. State the limit definition of $\int_a^i nfinityf(x)dx$.
5. Explain why $\int\_ infinity^i nfinityf(x)dx$ must usually be split into two pieces.
6. Give one example of a singularity at an interior point.

Core skill: direct computation and classification

1. Evaluate $\int_1^i nfinity1/x^2dx$.
2. Determine whether $\int_1^i nfinity1/xdx$ converges.
3. Evaluate $\int_0^1 1/\sqrt{x}dx$.
4. Determine whether $\int_0^1 1/xdx$ converges.
5. Determine whether $\int_2^i nfinity1/x^3dx$ converges.
6. Determine whether $\int_0^1 1/x^(3/4)dx$ converges.
7. Determine whether $\int_0^1 1/x^(5/4)dx$ converges.
8. Determine whether $\int_1^i nfinitye^{-x}dx$ converges.
9. Determine whether $\int\_ infinity^i nfinitye^{-x^2}dx$ should be approached by splitting the interval.

Interpretation: `p`-integrals, comparison, and tails

1. State the `p`-integral threshold on `[1, infinity)`.
2. State the `p`-integral threshold on `(0, 1]`.
3. Explain why the thresholds differ.
4. Use comparison to classify $\int_1^i nfinity1/(x^2+1)dx$.
5. Use comparison to classify $\int_1^i nfinity1/(x+3)dx$.
6. Explain in words what it means for a tail to contribute "arbitrarily little."
7. Compare the long-tail behavior of `1/x`, $1/x^2$, and e^{-x}.

Core skill: integral test and remainder language

1. Use the integral test to classify $sum 1/n^2$.

2. Use the integral test to classify $sum 1/n$.
3. Explain why positivity and monotonicity matter in the integral test.
4. Use an improper integral to bound the tail of $sum 1/n^2$.
5. Find an `N` that guarantees the tail of $sum 1/n^2$ is less than `0.01`.
6. Write a short paragraph connecting continuous area to the staircase picture of a positive decreasing series.

Challenge: subtle setup and reasoning

1. Explain why $\int\_ infinity^{i}nfinity x/(1 + x^2)dx$ is not automatically convergent just because the integrand is odd.
2. Give an example of a positive function that approaches `0` but whose improper integral diverges.
3. Give an example of a positive function that blows up at `0` but whose improper integral converges.
4. Describe one way a comparison argument can go wrong if the inequalities point in the wrong direction.
5. Explain why convergence of an improper integral is a global statement built from local tail or singular behavior.
6. Write a short note comparing convergence tests for improper integrals with convergence tests for series.

Modeling: probability and long-tail meaning

1. Verify that $p(x) = \lambda e^{(} - \lambda x)$ integrates to `1` on `[0, infinity)`.
2. Explain why a density can have infinite support and still describe total probability `1`.
3. Write a short paragraph explaining how expected value becomes an improper integral.
4. Describe a real setting where long-tail behavior matters more than central behavior.
5. Explain why the gamma function is evidence that improper integrals are mathematically richer than a single chapter of area problems.
6. Create a one-page guide for deciding when an improper integral should be attacked directly, by comparison, or through the integral test.

Reflection

Improper integrals force a mature version of the accumulation idea. The local rule has not changed: integrate small contributions. What changes is the domain or the size of the contributions. The chapter therefore asks the student to think less like a calculator and more like an analyst:

- what happens in the tail,

- what happens near the singularity,
- and what simpler benchmark controls the behavior?

That analytical shift is one reason the topic occupies real space in large calculus texts.

Suggested Homework Sets

These assignments are arranged as chapter-by-chapter ladders. Within each chapter, start with the `Warm-up` group, move to `Core skill`, then assign `Interpretation`, `Challenge`, and `Modeling` as needed for the course level. Exercise numbers run continuously through each chapter even though the sets are grouped by purpose.

Chapter 1

- `Warm-up: quantities, units, and rate meaning`: 1, 2
- `Core skill: average-rate computation`: 3, 4, 5
- `Interpretation: function stories and graph language`: 6, 7
- `Challenge: interval behavior and counterexamples`: 8, 9

Chapter 2

- `Warm-up: limit language and one-sided logic`: 1, 2, 3
- `Core skill: evaluating algebraic limits`: 4, 5, 6, 7, 8
- `Interpretation: continuity and discontinuity`: 9, 10
- `Challenge: constructing examples`: 11, 12, 13
- `Modeling: threshold rules and rounded outputs`: 14

Chapter 3

- `Warm-up: tangent lines and instantaneous rate meaning`: 1, 2, 3
- `Core skill: limit-definition practice`: 4, 5, 6, 7
- `Interpretation: local meaning and nondifferentiability`: 8, 9
- `Challenge: subtle derivative logic`: 10, 11, 12
- `Modeling: motion from sampled change`: 13

Chapter 4

- `Warm-up: basic derivative rules`: 1, 2, 3
- `Core skill: products, quotients, and compositions`: 4, 5, 6, 7, 8, 9
- `Interpretation: structure and local approximation`: 10, 11
- `Challenge: implicit and mixed structures`: 12, 13, 14

- `Modeling: chain-rule rates in context`: 15

Chapter 5

- `Warm-up: sign and critical-point meaning`: 1, 2, 3
- `Core skill: first- and second-derivative analysis`: 4, 5, 6, 7, 8
- `Interpretation: graph behavior from derivative evidence`: 9, 10
- `Challenge: subtle extrema and inflection logic`: 11, 12, 13
- `Modeling: optimization`: 14, 15

Chapter 6

- `Warm-up: integral meaning and units`: 1, 2, 3
- `Core skill: Riemann sums and basic definite integrals`: 4, 5, 6, 7, 8
- `Interpretation: signed accumulation and average value`: 9, 10
- `Challenge: net-change subtleties`: 11, 12, 13
- `Modeling: rate-to-total problems`: 14, 15

Chapter 7

- `Warm-up: antiderivatives and FTC meaning`: 1, 2, 3
- `Core skill: antiderivatives and FTC computation`: 4, 5, 6, 7, 8
- `Interpretation: derivatives and integrals together`: 9, 10
- `Challenge: accumulation inside compositions`: 11, 12, 13
- `Modeling: motion and reservoir totals`: 14, 15

Chapter 8

- `Warm-up: choosing an integration tool`: 1, 2, 3
- `Core skill: parts, substitution, and fractions`: 4, 5, 6, 7, 8
- `Interpretation: method choice and approximation`: 9, 10
- `Challenge: nonroutine integrals and error reasoning`: 11, 12, 13
- `Modeling: data-driven and physical integrals`: 14, 15

Chapter 9

- `Warm-up: geometric meaning of application formulas`: 1, 2, 3
- `Core skill: area, volume, mass, and work`: 4, 5, 6, 7

- `Interpretation: shell, slice, and density reasoning`: 8, 9
- `Challenge: setup choice and model validity`: 10, 11, 12
- `Modeling: volume and density integrals`: 13, 14

Chapter 10

- `Warm-up: sequence and series language`: 1, 2, 3
- `Core skill: convergence and geometric sums`: 4, 5, 6, 7, 8
- `Interpretation: sequence-series distinction and local approximation`: 9, 10
- `Challenge: counterexamples and divergence logic`: 11, 12, 13
- `Modeling: rebound totals and approximation`: 14, 15
- `Core skill: convergence tests`: 16, 17, 18, 19, 20, 21, 22
- `Interpretation: test choice`: 23
- `Challenge: alternating and absolute convergence`: 24, 25, 26, 27, 28
- `Modeling: Taylor and power-series approximations`: 29, 30, 31, 32, 33, 34

Chapter 11

- `Warm-up: slope fields and separability`: 1, 2, 3
- `Core skill: basic differential-equation solving`: 4, 5, 6, 7
- `Interpretation: exponential and logistic behavior`: 8, 9
- `Challenge: equilibrium and qualitative analysis`: 10, 11, 12
- `Modeling: cooling and population`: 13, 14
- `Core skill: autonomous equations and phase lines`: 15, 16, 17, 18
- `Interpretation: cooling and decay structure`: 19, 20, 21, 22, 23
- `Challenge: numerical methods and trust`: 24, 25, 26, 27, 28

Chapter 12

- `Warm-up: vectors and motion`: 1, 2, 3
- `Core skill: vector computation and curve derivatives`: 4, 5, 6, 7
- `Interpretation: speed, velocity, and geometry`: 8, 9
- `Challenge: orthogonality and path reasoning`: 10, 11, 12
- `Modeling: motion and work`: 13, 14

Chapter 13

- `Warm-up: surfaces, contours, and partial derivatives`: 1, 2, 3, 4, 5, 6

- `Core skill: gradients, tangent planes, and directional derivatives`: 7, 8, 9, 10, 11, 12
- `Interpretation: multivariable limits and local shape`: 13, 14, 15, 16, 17
- `Challenge: critical points and classification`: 18, 19, 20, 21, 22, 23
- `Modeling: constrained optimization and synthesis`: 24, 25, 26, 27, 28, 29, 30, 31, 32

Chapter 14

- `Warm-up: double and triple integral meaning`: 1, 2, 3
- `Core skill: iterated integrals and polar setup`: 4, 5, 6
- `Interpretation: geometry of repeated integration`: 7, 8
- `Challenge: coordinate choice and correction factors`: 9, 10, 11
- `Modeling: mass and concentration`: 12, 13

Chapter 15

- `Warm-up: vector fields and integral meanings`: 1, 2, 3
- `Core skill: describing fields and big-theorem ideas`: 4, 5, 6
- `Interpretation: circulation, flux, and boundary thinking`: 7, 8
- `Challenge: orientation and theorem philosophy`: 9, 10, 11
- `Modeling: divergence and work`: 12, 13

Chapter 16

- `Warm-up: parametrization and polar description`: 1, 2, 3, 4, 5, 6
- `Core skill: slopes, tangents, and local behavior`: 7, 8, 9, 10, 11, 12, 13, 14, 15
- `Interpretation: arc length, area, and geometry`: 16, 17, 18, 19, 20, 21
- `Challenge: curvature and nonroutine reasoning`: 31, 32, 33, 34, 35, 36
- `Modeling: motion, design, and coordinate choice`: 37, 38, 39, 40

Chapter 17

- `Warm-up: reading second-order models`: 1, 2, 3, 4, 5, 6
- `Core skill: characteristic equations and exact solutions`: 7, 8, 9, 10, 11, 12, 13, 14, 15
- `Interpretation: oscillation, decay, and behavior`: 16, 17, 18, 19, 20, 21, 22

- `Challenge: boundary values, modes, and reasoning`: 23, 24, 25, 26, 27, 28
- `Modeling: forcing, resonance, and computation`: 29, 30, 31, 32, 33, 34

Chapter 18

- `Warm-up: identifying improper structure`: 1, 2, 3, 4, 5, 6
- `Core skill: direct computation and classification`: 7, 8, 9, 10, 11, 12, 13, 14, 15
- `Interpretation: p-integrals, comparison, and tails`: 16, 17, 18, 19, 20, 21, 22
- `Challenge: subtle setup and reasoning`: 29, 30, 31, 32, 33, 34
- `Modeling: probability and long-tail meaning`: 35, 36, 37, 38, 39, 40

Assignment patterns

- `Foundational daily work`: assign `Warm-up` plus `Core skill`.
- `Discussion-centered homework`: assign `Core skill` plus `Interpretation`.
- `Stronger honors or proof-oriented set`: assign one or two `Challenge` items in addition to the core.
- `Applied course set`: assign `Warm-up`, selected `Core skill`, and the `Modeling` item.

Study Guide: Tactics, Tips, and Fun Facts

This section is not a replacement for the chapters. It is a compact set of cues for what to notice, what to try first, and what experienced students often learn too late.

1. Diagnose the problem before computing

Ask what kind of quantity is being discussed.

- If the question is about *how fast*, *how steep*, or *how sensitive*, think `derivative`.
- If the question is about a *total built from many small pieces*, think `integral`.
- If the question asks for *largest*, *smallest*, or *best*, think `optimization`.
- If the question asks what happens after *repeated steps* or *many terms*, think `sequence`, `series`, or `differential equation`.

Many calculus mistakes happen because students start differentiating or integrating before naming the structure of the problem.

2. The four habits that rescue most work

- Write the units. If the units do not make sense, the mathematics usually does not either.
- Sketch a picture, even a crude one. Shape often reveals sign, growth, and likely errors.
- Check extreme or simple cases such as $x = 0$, $x = 1$, or an endpoint.
- Say in words what the final answer means. A correct formula with no interpretation is usually unfinished work.

3. Derivative method cues

- A derivative should have units of `(output units)/(input units)`.
- If a function is built from layers, look for the `chain rule`.
- If a quantity is defined implicitly, differentiate every term and solve for the desired rate.
- If you need local behavior near one point, compute the derivative and then ask about sign, not just value.

Quick derivative check

When you finish differentiating, ask:

- Is the sign plausible?
- Does the result simplify at the point of interest?
- If the original function was increasing rapidly, does the derivative reflect that?

4. Integral method cues

- Try to name the tiny piece being added: `(rate)(time)`, `(density)(length)`, `(area)(thickness)`, or `(value)(width)`.
- If the integrand looks like a composition with its inside derivative present, try `substitution`.
- If the integrand looks like a product of algebraic and exponential or trigonometric pieces, try `integration by parts`.
- If the integrand is rational, check whether `partial fractions` is the right tool.
- If the data come from a table or a graph rather than a clean formula, use a `numerical method`.

Quick integral check

Ask whether the problem wants:

- net change or total amount,
- signed area or geometric area,
- an exact symbolic answer or a reliable estimate,
- and whether the variable of integration matches the slices or shells you chose.

5. Series and approximation cues

- A sequence is a list of terms; a series is a list of partial sums built from those terms.
- The term test can prove divergence, but it cannot prove convergence by itself.
- For a geometric series, the first thing to inspect is the ratio `r`.
- For a Taylor approximation, always say *near which point* the approximation is meant to work.

6. Multivariable and vector cues

- In several variables, one path is never enough evidence for a limit.

- The gradient points in the direction of steepest increase and is perpendicular to level curves.
- A tangent plane is just the local linear model in two input directions at once.
- In vector calculus, keep asking whether the theorem relates *interior behavior* to *boundary behavior*.

7. Fun facts worth remembering

- The word *calculus* comes from a Latin word for a small pebble used in counting.
- Much of early calculus grew out of two stubborn questions: how fast something is changing, and how much has accumulated.
- Power series are one of calculus' most surprising tools: near a point, complicated functions can behave like infinite polynomials.
- The major integral theorems all tell versions of the same story: local information inside a region can often be read from what happens on its boundary.

8. How to use the exercise ladders

Each chapter's exercises are grouped in this order:

- `Warm-up`: vocabulary, meaning, and first checks.
- `Core skill`: direct computation or setup practice.
- `Interpretation`: explanation, graph-reading, or translation tasks.
- `Challenge`: subtle cases, counterexamples, or deeper reasoning.
- `Modeling`: real-world setup and interpretation.

If you are stuck, step back one rung. If a `Challenge` problem will not move, redo one `Core skill` problem and one `Interpretation` problem first.

Appendix G. Chapter Review Sheets

This appendix is a fast-review companion for the full book. Each chapter sheet is intentionally compact enough to use before an exam, during office hours, or while designing an assignment sequence. The goal is not to replace the chapter, but to provide a high-density map of its major ideas, standard moves, and recurring mistakes.

How to use these sheets

Each chapter review has four parts:

- `Core ideas`: the concepts that hold the chapter together,
- `Standard moves`: the usual solution patterns,
- `Common mistakes`: what repeatedly goes wrong,
- `Oral-check questions`: quick prompts that reveal whether the main ideas are understood.

These are useful for both students and instructors because calculus often becomes clearer when its main structures are compressed into one page of questions and cues.

Chapter 1. Quantities, Functions, and the Shape of Change

`Core ideas`

- Calculus begins with varying quantities, not formulas alone.
- A function can be represented by formulas, tables, graphs, or words.
- Average rate of change compares total output change with total input change.
- Local linearity is the habit of replacing a complicated change process by a nearby linear one.

`Standard moves`

- Identify inputs, outputs, and units before computing.
- Translate verbal descriptions into functional relationships.
- Compute average rate over a stated interval and interpret the units.

`Common mistakes`

- Ignoring units.
- Treating a table or graph as less "real" than a formula.
- Confusing total change with rate of change.

`Oral-check questions`

- What is the difference between a quantity and a rate?
- Why is average rate an interval idea rather than a point idea?

Chapter 2. Nearness, Limits, and Continuity

`Core ideas`

- A limit describes nearby behavior, not necessarily function value.
- One-sided limits matter when the behavior differs by direction.
- Continuity means value and nearby behavior agree.
- Pathologies often come from holes, jumps, and mismatched rules.

`Standard moves`

- Estimate from tables and graphs before simplifying algebraically.
- Factor or rationalize when direct substitution gives an indeterminate form.
- Compare left-hand and right-hand behavior explicitly.

`Common mistakes`

- Confusing $f(a)$ with $\lim_{x \to a} f(x)$.
- Forgetting one-sided analysis for piecewise functions.
- Assuming continuity because the formula "looks nice."

`Oral-check questions`

- How can a limit exist when the function value is undefined?
- Why do two different one-sided limits destroy a two-sided limit?

Chapter 3. Derivatives

`Core ideas`

- The derivative is the limit of secant slopes.
- It records instantaneous rate of change and tangent-line slope.
- Differentiability is stronger than continuity.

- Higher derivatives describe how rates themselves change.

`Standard moves`

- Build difference quotients carefully.
- Interpret derivative values in context: slope, velocity, marginal change.
- Use higher derivatives to distinguish velocity from acceleration.

`Common mistakes`

- Dropping the limit too early.
- Treating the derivative as just a symbolic output.
- Forgetting that corners and cusps can destroy differentiability.

`Oral-check questions`

- Why is the derivative a function rather than just a number?
- What does a negative derivative mean physically?

Chapter 4. Working with Derivatives

`Core ideas`

- Derivative rules compress repeated limit arguments.
- The chain rule handles nested dependence.
- Implicit differentiation keeps derivative logic available when `y` is not isolated.
- Linearization turns derivatives into approximations.

`Standard moves`

- Read structure before differentiating.
- Keep products and quotients visibly grouped until the rule is complete.
- Use tangent-line approximations only near the base point.

`Common mistakes`

- Forgetting the inner derivative.
- Differentiating `y`-terms implicitly without attaching `y'`.
- Using linearization too far from the chosen anchor point.

`Oral-check questions`

- Why is the chain rule about composition rather than multiplication?
- When is logarithmic differentiation helpful?

Chapter 5. What Derivatives Tell Us

`Core ideas`

- The sign of `f'` controls increasing and decreasing behavior.
- Critical points are candidates, not automatic extrema.
- The sign of `f''` controls concavity.
- Optimization is derivative interpretation in context.

`Standard moves`

- Build sign charts for `f'` and `f''`.
- Use the first derivative test when interval behavior matters.
- Use the second derivative test when it is clean and conclusive.
- Check endpoints in closed-interval optimization.

`Common mistakes`

- Declaring extrema from $f'(c) = 0$ alone.
- Calling every zero of `f''` an inflection point.
- Optimizing the wrong quantity.

`Oral-check questions`

- Why is a sign change stronger evidence than a derivative value at one point?
- What makes optimization hard before the derivative is ever taken?

Chapter 6. The Integral as Accumulation

`Core ideas`

- Definite integrals represent total accumulation.
- Signed area and net change are closely related.
- Riemann sums approximate totals with many local pieces.
- Average value over an interval comes from accumulated total divided by interval length.

`Standard moves`

- Identify the local contribution and its sign.
- Interpret a negative integrand in context before integrating mechanically.
- Use Riemann-sum language to explain where the definite integral comes from.

`Common mistakes`

- Treating every integral as ordinary geometric area.

- Ignoring units in a rate-to-total problem.
- Confusing net change with total distance or total variation.

`Oral-check questions`

- Why can a definite integral be negative?
- What does the average value formula mean physically?

Chapter 7. Antiderivatives and the Fundamental Theorem

`Core ideas`

- Antiderivatives are families of functions.
- Initial conditions choose one member of the family.
- The Fundamental Theorem links local rates and accumulated totals.
- Substitution is the antidifferentiation partner of the chain rule.

`Standard moves`

- Separate indefinite-integral thinking from definite-integral thinking.
- Use units when explaining the Fundamental Theorem.
- Change bounds when substituting in definite integrals.

`Common mistakes`

- Forgetting `+ C`.
- Adding `+ C` to definite integrals.
- Treating substitution like guessing instead of structure recognition.

`Oral-check questions`

- What information is lost by differentiation and recovered by antidifferentiation?
- Why does an accumulation function behave like a new function?

Chapter 8. More Integration Tools

`Core ideas`

- Integration is partly method choice.
- Integration by parts reverses the product rule.
- Trigonometric methods use structural identities.
- Partial fractions rewrite rational expressions into standard pieces.

- Numerical integration matters when exact antiderivatives are unavailable or unnecessary.

`Standard moves`

- Diagnose the integrand before choosing a method.
- Differentiate one factor and integrate the other in parts.
- Factor denominators before deciding whether partial fractions applies.
- Use numerical methods for data or awkward exact inputs.

`Common mistakes`

- Using a favorite technique on every problem.
- Forgetting absolute values after logarithmic antiderivatives.
- Reporting numerical estimates as exact values.

`Oral-check questions`

- Why is method choice the real challenge of Chapter 8?
- When is a numerical answer more appropriate than a symbolic one?

Chapter 9. Applications of Integration

`Core ideas`

- Application formulas all come from local pieces.
- Area, volume, work, mass, and fluid force share a slicing blueprint.
- Washer and shell methods are geometric choices, not unrelated formulas.
- Units are part of the setup, not decoration.

`Standard moves`

- Draw one representative slice before integrating.
- Choose dx or dy according to the cleanest geometry.
- Decide whether slices or shells are more natural before writing the integral.

`Common mistakes`

- Subtracting in the wrong order.
- Choosing a cumbersome slice direction.
- Forgetting what the integrand means physically.

`Oral-check questions`

- What does one thin slice contribute in an application problem?

- Why are units a setup check?

Chapter 10. Sequences and Series

`Core ideas`

- Sequences study term behavior; series study partial-sum behavior.
- Term limits are necessary but not sufficient for convergence.
- Geometric behavior is a foundational benchmark.
- Taylor polynomials use local derivative data to build approximations.

`Standard moves`

- Keep the sequence/series distinction explicit.
- Compare unknown series to known tail behavior.
- State the center point before using a Taylor approximation.

`Common mistakes`

- Confusing terms with partial sums.
- Assuming terms going to zero guarantees convergence.
- Using a Taylor polynomial far from its center without comment.

`Oral-check questions`

- Why is an infinite series really a limit of finite sums?
- What does "local" mean in Taylor approximation?

Chapter 11. Differential Equations and Models

`Core ideas`

- Differential equations are rate laws.
- Slope fields provide qualitative information before explicit solutions are known.
- Separable equations can often be solved by organizing variables and integrating.
- Exponential and logistic models show how local rules generate global behavior.
- Numerical methods extend the subject beyond symbolic solvability.

`Standard moves`

- Read the equation qualitatively first.
- Separate variables when possible.
- Use initial conditions after the general family is found.

- Track equilibria and sign behavior in logistic-style models.

`Common mistakes`

- Treating the equation as pure algebra rather than a model.
- Forgetting the initial condition.
- Assuming exponential growth remains realistic forever.

`Oral-check questions`

- What does a slope field reveal that a formula may hide?
- Why do equilibrium values matter so much?

Chapter 12. Vectors and Space

`Core ideas`

- Vectors separate direction from location.
- Lines use direction vectors; planes use normal vectors.
- Parametric curves describe motion naturally.
- Dot products measure angle and projection.
- Cross products measure oriented area and build normals.

`Standard moves`

- Normalize a vector when only direction matters.
- Write lines as point plus direction.
- Write planes from point plus normal.
- Interpret velocity as a vector and speed as its magnitude.

`Common mistakes`

- Mixing up points and vectors.
- Confusing speed with velocity.
- Using dot and cross products interchangeably.

`Oral-check questions`

- Why does a plane need a normal vector?
- What geometric information does a cross product contain?

Chapter 13. Multivariable Functions

`Core ideas`

- Surfaces generalize graphs, and contour maps compress them.
- Multivariable limits are path-sensitive.
- Partial derivatives freeze some variables while allowing one to move.
- The gradient points in the direction of steepest increase.
- Tangent planes provide local linear models.

`Standard moves`

- Test path dependence early when disproving a limit.
- State which variables are held fixed in a partial derivative.
- Interpret the gradient geometrically, not just computationally.

`Common mistakes`

- Assuming a few matching paths prove a limit exists.
- Forgetting what is held constant in a partial derivative.
- Treating a tangent plane as globally reliable.

`Oral-check questions`

- Why are multivariable limits subtler than one-variable limits?
- How does the gradient summarize directional change?

Chapter 14. Multiple Integration

`Core ideas`

- Double and triple integrals accumulate over regions and solids.
- Iterated integrals organize the accumulation in stages.
- Polar coordinates match radial geometry.
- Change of variables compensates for geometric distortion.

`Standard moves`

- Sketch the region first.
- Reverse the order of integration if the bounds simplify.
- Use polar coordinates when circular symmetry dominates.

`Common mistakes`

- Dropping the `r` factor in polar coordinates.
- Writing bounds that do not match the region.
- Treating `dA` or `dV` like a disposable symbol.

`Oral-check questions`

- Why does coordinate choice matter so much?
- What is the geometric meaning of a Jacobian-like correction factor?

Chapter 15. Vector Calculus

`Core ideas`

- Vector fields assign directional data across space.
- Line integrals accumulate along paths.
- Surface integrals accumulate across oriented surfaces.
- Flux and circulation are different kinds of accumulation.
- The great integral theorems connect interior behavior with boundary totals.

`Standard moves`

- Decide whether the problem is about motion along a path or flow across a boundary.
- Check orientation before interpreting the sign.
- Compare Green, Divergence, and Stokes as parallel local-to-global patterns.

`Common mistakes`

- Confusing flux with circulation.
- Ignoring orientation.
- Memorizing theorem names without seeing the common structure.

`Oral-check questions`

- How does Green's Theorem echo the Fundamental Theorem of Calculus?
- Why are the major integral theorems best understood as one family?

Chapter 16. Parametric Curves, Polar Coordinates, and Curvature

`Core ideas`

- Parametrization tracks order, direction, and repeated tracing.
- Polar coordinates reorganize planar geometry around radius and angle.
- Arc length and area formulas arise from local geometric pieces.
- Curvature measures turning rather than mere tilt.

`Standard moves`

- Compute the point and the slope separately for tangent-line questions.

- Keep the parameter until motion information is no longer needed.
- Use symmetry and zeros of `r` before plotting many polar sample points.
- Check special cases like circles when learning a new formula.

`Common mistakes`

- Treating dy/dt as the plane slope without dividing by dx/dt.
- Forgetting that polar coordinates are not unique.
- Mixing the polar area and arc-length formulas.
- Talking about curvature after computing only slope.

`Oral-check questions`

- Why can a parametrized curve represent something that is not the graph of a function?
- What does curvature measure that slope cannot?

Chapter 17. Second-Order Differential Equations and Oscillation

`Core ideas`

- Second-order equations encode acceleration laws.
- Characteristic roots control the visible behavior of solutions.
- Harmonic motion, damping, and forcing are model families rather than isolated tricks.
- Initial-value and boundary-value problems ask different questions.

`Standard moves`

- Translate words into initial or boundary conditions before solving.
- Use the characteristic equation to classify the solution family.
- Interpret period, decay, or oscillation after solving for constants.
- Distinguish transient response from persistent forcing.

`Common mistakes`

- Forgetting that two initial conditions are usually needed.
- Omitting the factor `t` in the repeated-root case.
- Confusing period with angular frequency.
- Solving algebraically without interpreting the physical behavior.

`Oral-check questions`

- Why do complex roots correspond to oscillation?
- What role does damping play in whether a system keeps crossing equilibrium?

Chapter 18. Improper Integrals and Long-Tail Behavior

`Core ideas`

- Improper integrals are limits of ordinary integrals.
- Convergence depends on tail behavior or singular behavior.
- Comparison with benchmark integrands is the main strategic tool.
- Improper integrals link continuous accumulation to series and probability.

`Standard moves`

- Rewrite the integral as a limit before integrating.
- Split at every singular point or infinite endpoint.
- Compare with $1/x^p$ when the behavior is power-like.
- Use the integral test when a positive decreasing series is attached to the problem.

`Common mistakes`

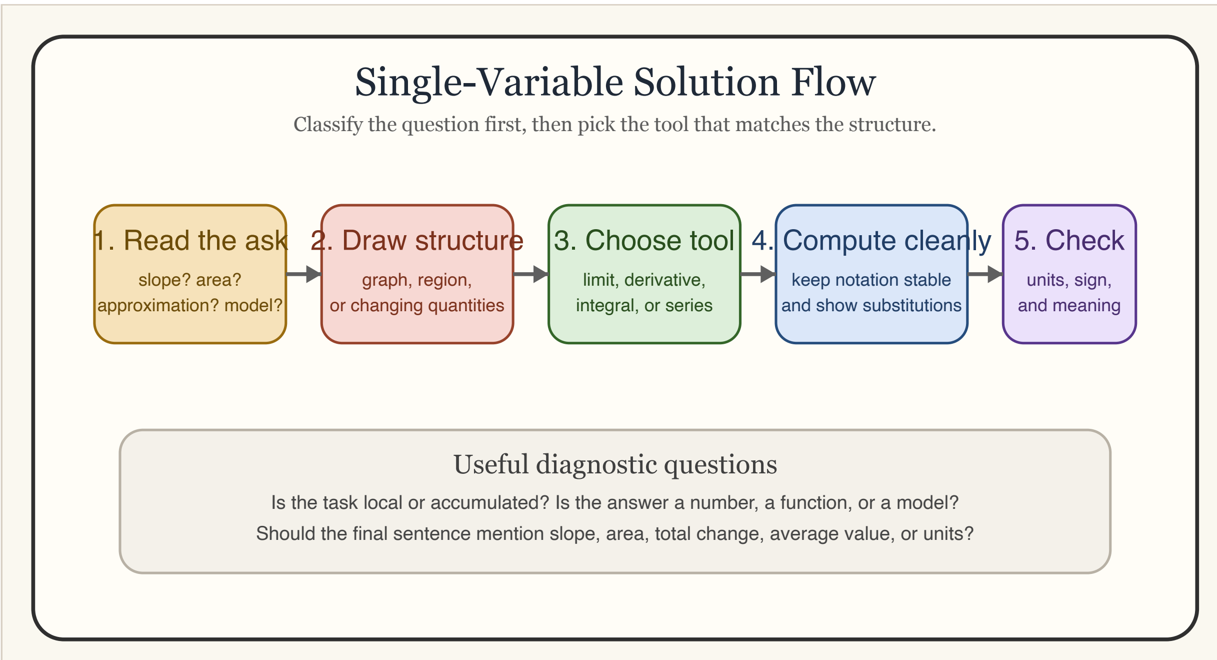
- Treating an improper integral as proper and forgetting the limit.
- Failing to split at an interior singularity.
- Using comparison in the wrong inequality direction.
- Assuming any function that tends to zero must have finite total area.

`Oral-check questions`

- Why can an infinite region still have finite area?
- What does it mean for the tail to contribute arbitrarily little?

Appendix O. Guided Examples for Single-Variable Calculus

This appendix collects longer worked examples for the single-variable portion of the book. The goal is not only to show correct solutions, but to model the sequence of choices an experienced student or instructor makes while solving. Each example is intentionally written in a stepwise style so it can be used for tutoring, board-work preparation, or independent review.



Example 1. Average rate of change from a data table

A fish population in a lake is estimated to be `12,400` in year `2` and `15,100` in year `5`. Find the average rate of change over the interval from year `2` to year `5` and interpret the answer.

Step 1: identify output change

The output changes by

$15,100 - 12,400 = 2,700.$

Step 2: identify input change

The input changes by

$5 - 2 = 3$ years.

Step 3: divide output change by input change

The average rate of change is

$2,700/3 = 900$.

Interpretation

Over this interval, the population increased by an average of `900` fish per year.

Why this example matters

The arithmetic is simple, but the habit matters. Average rate of change is always "change in output divided by change in input," and the units belong in the final sentence.

Example 2. A removable discontinuity hidden by algebra

Evaluate

$\lim_{(} x \to 2)(x^2 - 4)/(x - 2)$.

Step 1: test direct substitution carefully

Substituting $x = 2$ gives

$(4 - 4)/(2 - 2) = 0/0$,

which is indeterminate. That means the original form does not reveal the limit directly.

Step 2: factor the numerator

$x^2 - 4 = (x - 2)(x + 2)$.

So for $x \neq 2$,

$(x^2 - 4)/(x - 2) = x + 2$.

Step 3: evaluate the simplified expression at the target value

Now the limit becomes

$\lim_{(x \to 2)}(x + 2) = 4.$

Interpretation

The graph behaves like the line $y = x + 2$ except for a hole at $x = 2$.

Common mistake

Students sometimes conclude that the function value at $x = 2$ is `4`. The problem asked for a limit, not a function value. The limit can exist even when the original rule is undefined at the point.

Example 3. Derivative from first principles

Use the limit definition to find the derivative of $f(x) = x^2$.

Step 1: write the difference quotient

$f'(x) = \lim_{(h \to 0)}[f(x + h) - f(x)]/h.$

Because $f(x) = x^2$,

$f'(x) = \lim_{(h \to 0)}[(x + h)^2 - x^2]/h.$

Step 2: expand

$(x + h)^2 = x^2 + 2xh + h^2.$

So

$f'(x) = \lim_{(h \to 0)}[x^2 + 2xh + h^2 - x^2]/h.$

Step 3: simplify before taking the limit

$f'(x) = \lim_{(h \to 0)}[2xh + h^2]/h = \lim_{(h \to 0)}(2x + h).$

Step 4: take the limit

$f'(x) = 2x.$

Why this example matters

The essential skill is not expansion alone. It is the discipline of simplifying the quotient before substituting $h = 0$.

Example 4. Chain rule and linear approximation together

Find the derivative of $f(x) = \sqrt{1 + 3x^2}$ and then use linearization at $x = 0$ to estimate $\sqrt{1.12}$.

Step 1: differentiate by identifying inner and outer structure

The outer function is $\sqrt{u}$ and the inner function is $u = 1 + 3x^2$.

So

$f'(x) = [1/(2\sqrt{1 + 3x^2})] * 6x = 3x/\sqrt{1 + 3x^2}$.

Step 2: build the linearization at $x = 0$

We need $f(0)$ and `f'(0)`.

- $f(0) = 1$
- $f'(0) = 0$

So the linearization is

$L(x) = 1$.

Step 3: match the target number to the model

We want $\sqrt{1.12}$, so solve

$1 + 3x^2 = 1.12$.

Then

$3x^2 = 0.12$, so $x^2 = 0.04$, and $x = 0.2$ or $x = -0.2$.

Step 4: evaluate the linear model

$\sqrt{1.12} = f(0.2) \approx L(0.2) = 1$.

Interpretation

This estimate is crude because the slope at the base point is zero, but the example shows the workflow: identify the function, choose a nearby point, linearize, then translate the original quantity into the model variable.

Example 5. Optimization with a fixed perimeter

A rectangle has perimeter `80`. Find the dimensions that maximize its area.

Step 1: define variables

Let the side lengths be `x` and `y`.

The perimeter condition gives

$2x + 2y = 80,$

so

$y = 40 - x.$

Step 2: write the quantity to optimize

Area is

$A = xy = x(40 - x) = 40x - x^2.$

Step 3: differentiate

$A'(x) = 40 - 2x.$

Step 4: find critical points

Set $A'(x) = 0$:

$40 - 2x = 0$, so $x = 20$.

Then

$y = 20.$

Step 5: interpret

The area is maximized by a square of side `20`.

Why this example matters

Optimization problems become manageable when the geometry is converted into one variable before differentiating.

Example 6. A Riemann-sum viewpoint on accumulation

A velocity function is $v(t) = 2t + 1$ for $0 \leq t \leq 3$. Find the total displacement over this interval.

Step 1: identify the accumulated quantity

Displacement is the integral of velocity:

$$\int_0^3 (2t + 1)dt.$$

Step 2: antiderivative

An antiderivative is

$$t^2 + t.$$

Step 3: apply the bounds

$$[t^2 + t]_0^3 = (9 + 3) - 0 = 12.$$

Interpretation

The total displacement is `12` units.

Riemann-sum interpretation

The same answer could be seen as the limit of adding many short rectangles with height equal to velocity on tiny time intervals. The definite integral records the limit of those local contributions.

Example 7. The Fundamental Theorem in action

Let

$F(x) = \int_1^x (t^2 + 3)dt.$

Find `F'(x)` and compute $F(2)$.

Step 1: derivative

By the Fundamental Theorem of Calculus,

$F'(x) = x^2 + 3.$

Step 2: evaluate the definite integral

An antiderivative of $t^2 + 3$ is

$t^3/3 + 3t.$

So

$F(2) = [t^3/3 + 3t]_1^2 = (8/3 + 6) - (1/3 + 3) = 16/3.$

Why this example matters

This example shows the two central jobs of the theorem:

- it differentiates accumulation functions,
- and it evaluates definite integrals through antiderivatives.

Example 8. Choosing an integration method

Evaluate

$int \times e^x dx.$

Step 1: recognize the structure

This is a product of an algebraic factor and an exponential factor. Integration by parts is the natural choice.

Step 2: choose `u` and `dv`

Let

- $u = x$, so $du = dx$,
- $dv = e^x dx$, so $v = e^x$.

Step 3: apply the formula

$$intudv = uv - intvdu.$$

So

$$int \times e^x dx = xe^x - inte^x dx = xe^x - e^x + C.$$

Step 4: factor if helpful

$$int \times e^x dx = e^x (x - 1) + C.$$

Common mistake

Students often choose $u = e^x$, which does not simplify when differentiated. The "LIATE"-style heuristic is less important than the underlying simplification goal.

Example 9. Area between curves

Find the area between $y = x$ and $y = x^2$ on `[0,1]`.

Step 1: determine which curve is on top

On `[0,1]`, we have $x \geq x^2$, so the upper curve is $y = x$.

Step 2: set up the integral

Area is

$$\int_0^1 (x - x^2) dx.$$

Step 3: integrate

$$\int_0^1 (x - x^2) dx = [x^2/2 - x^3/3]_0^1 = 1/2 - 1/3 = 1/6.$$

Why this example matters

The entire setup depends on "top minus bottom." Most area-between-curves errors happen before the integral is even evaluated.

Example 10. Volume by shells

Rotate the region under $y = x$ on `[0,1]` about the `y`-axis. Find the resulting volume using cylindrical shells.

Step 1: identify shell radius and height

At position `x`:

- radius = `x`,
- height = `x`.

Step 2: write the shell formula

$V = 2pi \int_0^1 (radius)(height) dx = 2pi \int_0^1 x * x dx = 2pi \int_0^1 x^2 dx.$

Step 3: evaluate

$V = 2pi [x^3/3]_0^1 = 2pi/3.$

Why this example matters

The shell method is often easier than washers when the axis of rotation and the given function interact awkwardly.

Example 11. Arc length as accumulated local distance

Find the arc length of $y = x^(3/2)$ from $x = 0$ to $x = 4$.

Step 1: recall the formula

$L = \int_a^b sqrt(1 + (y')^2) dx.$

Step 2: differentiate

$y' = (3/2) x^(1/2).$

So

$(y')^2 = 9x/4.$

Step 3: set up the integral

$L = \int_0^4 \sqrt{1 + 9x/4}dx.$

Step 4: use substitution

Let $u = 1 + 9x/4$. Then $du = 9/4dx$, so $dx = 4/9du$.

When $x = 0$, $u = 1$; when $x = 4$, $u = 10$.

So

$L = (4/9) \int_1^1 0u^(1/2)du = (4/9) * [(2/3)u^(3/2)]_1^1 0.$

Therefore

$L = (8/27)(10^(3/2) - 1).$

Interpretation

Arc length turns local slope information into distance. The formula is a direct generalization of Pythagorean reasoning on tiny line segments.

Example 12. Work done by a variable force

A spring force is $F(x) = 5x$ newtons for displacement `x` measured in meters. Find the work required to stretch the spring from $x = 0$ to $x = 3$.

Step 1: set up the work integral

$W = \int_0^3 F(x)dx = \int_0^3 5x dx.$

Step 2: evaluate

$W = [5x^2/2]_0^3 = 45/2.$

Interpretation

The work is `22.5` joules.

Why this example matters

Work problems are a model of the larger calculus habit: identify the local contribution first, then accumulate it.

Example 13. Implicit differentiation and a tangent line

Find the tangent line to the curve

$$x^2 + xy + y^2 = 7$$

at the point `(1, 2)`.

Step 1: check that the point lies on the curve

Substitute `(1, 2)`:

$1 + 2 + 4 = 7.$

So the point really is on the curve.

Step 2: differentiate both sides with respect to `x`

Differentiate term by term:

- derivative of x^2 is `2x`,
- derivative of `xy` requires the product rule, giving `x y' + y`,
- derivative of y^2 is `2y y'`.

So

$2x + xy' + y + 2yy' = 0.$

Step 3: solve for `y'`

Group the terms containing `y'`:

$(x + 2y)y' = -(2x + y).$

Therefore

$y' = -(2x + y)/(x + 2y).$

Step 4: evaluate the slope at `(1, 2)`

$y'(1,2) = -(2+2)/(1+4) = -4/5.$

Step 5: write the tangent line

Using point-slope form,

$y - 2 = (-4/5)(x - 1).$

Why this example matters

Implicit differentiation is not a new rule. It is the chain rule and product rule applied when `y` is a function of `x` even though the equation is not solved for `y`.

Example 14. A related-rates problem with changing volume

Air is pumped into a spherical balloon so that its radius increases at `0.3` centimeters per second. How fast is the volume increasing when the radius is `10` centimeters?

Step 1: start with the geometric formula

For a sphere,

$V = (4/3)\pi r^3.$

Step 2: differentiate with respect to time

Because both `V` and `r` change with time,

$dV/dt = 4\pi r^2 dr/dt.$

Step 3: substitute the known values

At the instant when $r = 10$ and $dr/dt = 0.3$,

$dV/dt = 4\pi(10)^2(0.3) = 120\pi.$

Conclusion

The volume is increasing at 120π cubic centimeters per second.

Common mistake

Students sometimes plug numbers into the volume formula before differentiating. Related-rates problems almost always work better when the formula is differentiated symbolically first.

Example 15. Using first and second derivatives together

Analyze the critical points and inflection points of

$f(x) = x^4 - 4x^2.$

Step 1: compute the first derivative

$f'(x) = 4x^3 - 8x = 4x(x^2 - 2).$

So the critical numbers are

- $x = 0,$
- $x = \sqrt{2},$
- $x = -\sqrt{2}.$

Step 2: compute the second derivative

$f''(x) = 12x^2 - 8.$

Step 3: classify the critical points

- At $x = 0$, $f''(0) = -8 < 0$, so there is a local maximum.
- At $x = +-\sqrt{2}$, $f'' = 16 > 0$, so there are local minima.

Step 4: find inflection points

Set $f''(x) = 0$:

$12x^2 - 8 = 0$, so $x^2 = 2/3$.

Thus the inflection points occur at

$x = +-\sqrt{2/3}.$

Interpretation

The graph has a hill at the origin, dips to valleys at $\pm\sqrt{2}$, and changes concavity at $\pm\sqrt{2/3}$.

Why this example matters

The first derivative detects where behavior can change. The second derivative refines the story by telling whether the graph bends up or down.

Example 16. Definite integration by substitution

Evaluate

$\int_0^1 2x\cos(x^2)dx.$

Step 1: identify the inner function

The expression x^2 sits inside the cosine, and its derivative is `2x`. That makes substitution natural.

Step 2: substitute

Let

$u = x^2$, so $du = 2x\,dx.$

Change the bounds:

- when $x = 0$, $u = 0$,
- when $x = 1$, $u = 1$.

Step 3: rewrite the integral

$\int_0^1 2x\cos(x^2)dx = \int_0^1 \cos u\,du.$

Step 4: integrate

$\int_0^1 \cos u\,du = [\sin u]_0^1 = \sin 1.$

Why this example matters

Keeping the definite-integral bounds in the new variable avoids the common habit of switching back at the end.

Example 17. Partial fractions with two linear factors

Evaluate

$int(3x + 5)/(x^2 + x - 2)dx.$

Step 1: factor the denominator

$x^2 + x - 2 = (x + 2)(x - 1).$

Step 2: set up the partial-fraction form

Assume

$(3x + 5)/[(x + 2)(x - 1)] = A/(x + 2) + B/(x - 1).$

Multiply through by `(x + 2)(x - 1)`:

$3x + 5 = A(x - 1) + B(x + 2).$

Step 3: match coefficients

Expand the right-hand side:

$3x + 5 = (A + B)x + (-A + 2B).$

So

- $A + B = 3,$
- $-A + 2B = 5.$

Solving gives $A = 1$ and $B = 2.$

Step 4: integrate term by term

$int(3x + 5)/(x^2 + x - 2)dx = int1/(x + 2)dx + int2/(x - 1)dx.$

Therefore

$= \ln|x+2| + 2\ln|x-1| + C.$

Common mistake

The logarithm of a denominator factor is correct only after the rational expression has been separated into pieces whose numerators match the derivative structure.

Example 18. Midpoint and trapezoidal approximations compared

Approximate

$$\int_0^2 x^2 dx$$

using both the midpoint rule and the trapezoidal rule with $n = 2$. Then compare each approximation with the exact value.

Step 1: compute the exact value for reference

$\int_0^2 x^2 dx = [x^3/3]_0^2 = 8/3.$

Step 2: midpoint rule

With $n = 2$, the width is $\Delta x = 1$. The midpoints are `0.5` and `1.5`.

So

$M_2 = 1[f(0.5) + f(1.5)] = 0.25 + 2.25 = 2.5.$

Step 3: trapezoidal rule

Using $x = 0, 1, 2$,

$T_2 = (1/2)[f(0) + 2f(1) + f(2)] = (1/2)(0 + 2 + 4) = 3.$

Step 4: compare

- exact value = $8/3 \approx 2.667$,
- midpoint approximation = `2.5`,
- trapezoidal approximation = `3`.

Interpretation

Because x^2 is concave up, midpoint rectangles tend to underestimate while trapezoids tend to overestimate.

Example 19. Average value and a representative input

Find the average value of $f(x) = x^2 + 1$ on `[0, 2]`, and then find a number `c` in `[0, 2]` for which $f(c)$ equals that average value.

Step 1: compute the average value

The average value on `[a, b]` is

$(1/(b - a)) \int_a^b f(x)dx.$

So here

$f_a vg = (1/2) \int_0^2 (x^2 + 1)dx.$

Step 2: integrate

$f_a vg = (1/2)[x^3/3 + x]_0^2 = (1/2)(8/3 + 2) = 7/3.$

Step 3: solve $f(c) = f_a vg$

We want

$c^2 + 1 = 7/3.$

So

$c^2 = 4/3$, giving $c = 2/\sqrt{3}$.

Because $2/\sqrt{3}$ lies in `[0, 2]`, it is a valid representative input.

Why this example matters

The average value of a function is not simply the midpoint height. It is the height of a constant function that gives the same accumulated area over the interval.

Example 20. Mass and center of mass of a rod

A thin rod lies on $0 \le x \le 3$ with density

$\varrho(x) = 1 + x.$

Find its mass and center of mass.

Step 1: compute the mass

Mass is density accumulated over the rod:

$m = \int_0^3 (1 + x)dx = [x + x^2/2]_0^3 = 3 + 9/2 = 15/2.$

Step 2: compute the first moment

For a rod on the `x`-axis,

$M = \int_0^3 x\varrho(x)dx = \int_0^3 x(1 + x)dx.$

So

$M = \int_0^3 (x + x^2)dx = [x^2/2 + x^3/3]_0^3 = 9/2 + 9 = 27/2.$

Step 3: divide moment by mass

$x_bar = M/m = (27/2)/(15/2) = 27/15 = 9/5.$

Conclusion

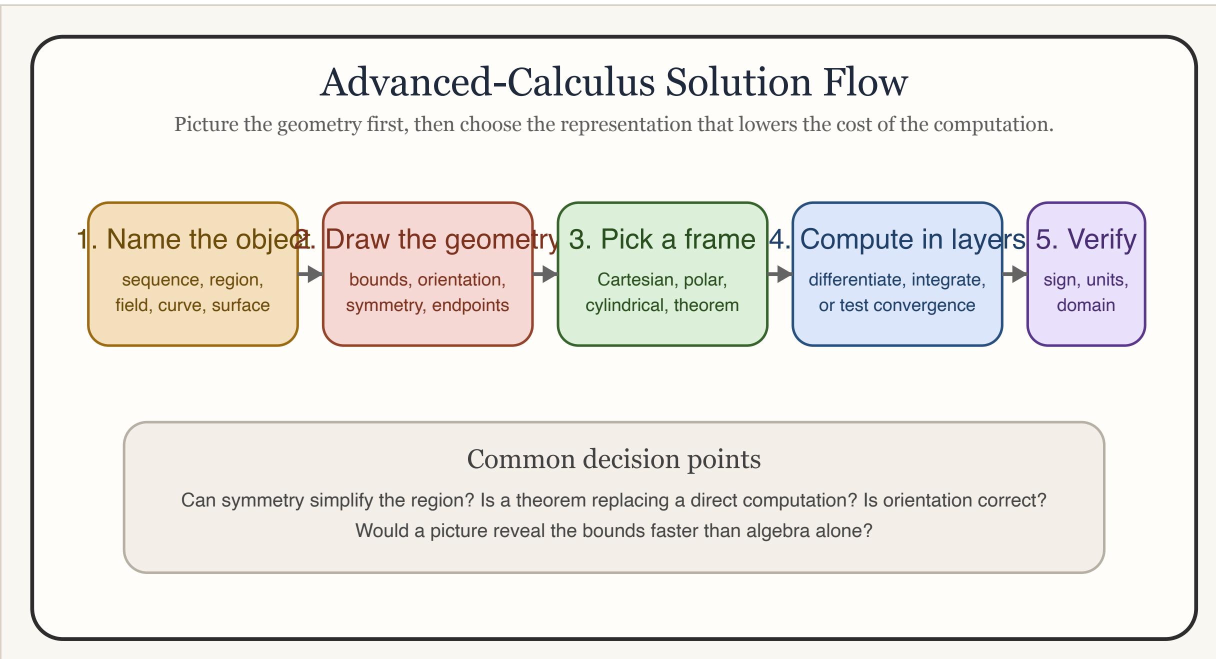
The rod has mass $15/2$, and its center of mass is at $x = 9/5$.

Interpretation

Since density increases as `x` increases, the balance point lies to the right of the midpoint $x = 1.5$.

Appendix P. Guided Examples for Series, Models, and Multivariable Calculus

This appendix continues the worked-example style of Appendix O, but focuses on the later parts of the book: sequences and series, differential equations, vectors, multivariable calculus, multiple integration, and vector calculus. The examples are written as dense review models rather than as terse answer-key snippets.



Example 1. Classifying a sequence and a related series

Consider the sequence $a_n = 1/n$ and the series $sum 1/n$.

Step 1: classify the sequence

Because $1/n \to 0$, the sequence converges to `0`.

Step 2: classify the series

The series

$$1 + 1/2 + 1/3 + ...$$

is the harmonic series, which diverges.

Lesson

The terms of a series can approach zero while the series itself still diverges. This is the first major distinction to keep mentally separate in Chapter 10.

Example 2. Ratio test with a factorial

Determine whether

$$sum n!/10^n$$

converges.

Step 1: compute the ratio

Let $a_n = n!/10^n$. Then

$a_(n + 1)/a_n = (n + 1)!/10^(n + 1) * 10^n/n! = (n + 1)/10.$

Step 2: take the limit

As $n \to infinity$, this ratio grows without bound. In particular it eventually exceeds `1`.

Conclusion

The series diverges.

Why this example matters

Factorial growth eventually dominates fixed exponential growth. The ratio test exposes that quickly.

Example 3. Alternating-series error estimate

Approximate

$$1 - 1/3 + 1/5 - 1/7 + ...$$

using the first four terms and bound the error.

Step 1: compute the partial sum

$S_4 = 1 - 1/3 + 1/5 - 1/7$.

Using a common denominator of `105`,

$S_4 = (105 - 35 + 21 - 15)/105 = 76/105$.

Step 2: identify the next omitted term

The next term has magnitude $1/9$.

Conclusion

The estimate is $76/105$, and the error is at most $1/9$.

Why this example matters

The alternating-series remainder theorem gives guaranteed accuracy with minimal work.

Example 4. Taylor approximation of e^x

Use the quadratic Maclaurin polynomial to approximate $e^0.3$.

Step 1: write the polynomial

For e^x,

$P_2(x) = 1 + x + x^2/2$.

Step 2: substitute

$P_2(0.3) = 1 + 0.3 + 0.09/2 = 1.345$.

Interpretation

The exact value is about `1.34986`, so the approximation is close and slightly low.

Why this example matters

Taylor polynomials are practical approximators, not only theoretical expansions.

Example 5. Logistic model and phase line

Analyze

$dP/dt = 0.4P(1 - P/500)$.

Step 1: identify equilibria

Set the derivative equal to zero:

$0.4P(1 - P/500) = 0$.

So $P = 0$ or $P = 500$.

Step 2: determine signs

- for $0 < P < 500$, both factors are positive, so $P' > 0$,
- for $P > 500$, the second factor is negative, so $P' < 0$.

Conclusion

Solutions move upward toward `500` when below it and downward toward `500` when above it. The carrying capacity `500` is stable.

Example 6. One Euler estimate and one interpretation check

Use Euler's Method with $h = 0.1$ to estimate $y(0.2)$ for

$y' = x + y, y(0) = 1$.

Step 1: first step

At `(0,1)`, slope = `1`.

So

$y(0.1) \approx 1 + 0.1(1) = 1.1$.

Step 2: second step

At `(0.1, 1.1)`, slope = `1.2`.

So

$y(0.2) \approx 1.1 + 0.1(1.2) = 1.22$.

Interpretation check

Because the slope is increasing over this early interval, the true curve should bend upward, so a piecewise linear estimate may fall slightly below the exact solution. That qualitative check strengthens confidence in the numerical result.

Example 7. Vector geometry in space

Find the angle between $u = \langle 1, 2, 2 \rangle$ and $v = \langle 2, 1, 2 \rangle$.

Step 1: compute the dot product

$u \cdot v = 1*2 + 2*1 + 2*2 = 8$.

Step 2: compute the magnitudes

$|u| = \sqrt{1 + 4 + 4} = 3$,

$|v| = \sqrt{4 + 1 + 4} = 3$.

Step 3: use the dot-product formula

$\cos \theta = (u \cdot v)/(|u||v|) = 8/9$.

So

$\theta = \arccos(8/9)$.

Why this example matters

The dot product turns geometric angle information into algebra.

Example 8. Directional derivative from a gradient

Let $f(x, y) = x^2y + y^2$. Find the directional derivative at `(1,2)` in the direction of $\langle 3, 4 \rangle$.

Step 1: compute the gradient

- $f_x = 2xy$

- $f_y = x^2 + 2y$

At `(1,2)`,

$\nabla f = \langle 4, 5 \rangle$.

Step 2: normalize the direction

The vector $\langle 3, 4 \rangle$ has length `5`, so the unit direction is

$u = \langle 3/5, 4/5 \rangle$.

Step 3: take the dot product

$D_u f(1, 2) = \langle 4, 5 \rangle . \langle 3/5, 4/5 \rangle = 12/5 + 20/5 = 32/5$.

Lesson

For directional derivatives, normalization is part of the method, not a cosmetic detail.

Example 9. Classifying a critical point

Classify the critical point of $f(x, y) = x^2 + y^2 - 6x + 2y$.

Step 1: find the critical point

- $f_x = 2x - 6$
- $f_y = 2y + 2$

Setting both equal to zero gives `(3, -1)`.

Step 2: compute the second derivatives

- $f_x x = 2$
- $f_y y = 2$
- $f_x y = 0$

So

$D = f_x x f_y y - (f_x y)^2 = 4$.

Step 3: apply the test

Because $D > 0$ and $f_x x > 0$, the point `(3, -1)` is a local minimum.

Example 10. Reversing the order of integration

Rewrite

$$\int_0^1 \int_x^1 f(x, y)\,dy\,dx$$

with the opposite order of integration.

Step 1: describe the region

The region satisfies:

- $0 \le x \le 1$
- $x \le y \le 1$

This is the triangular region above the line $y = x$ and below $y = 1$.

Step 2: sweep horizontally

For a fixed `y` between `0` and `1`, the variable `x` runs from `0` up to `y`.

Result

The reversed integral is

$\int_0^1 \int_0^y f(x, y)\,dx\,dy$.

Why this example matters

Changing order is a geometric decision. The algebra follows the picture.

Example 11. Polar coordinates for a disk integral

Evaluate

$$\iint_R (x^2 + y^2)\,dA$$

where `R` is the disk $x^2 + y^2 \le 4$.

Step 1: translate the integrand

In polar coordinates,

$x^2 + y^2 = r^2$.

Step 2: include the area factor

$dA = r dr dtheta$.

Step 3: write the bounds

For the disk of radius `2`,

- $0 \leq r \leq 2$
- $0 \leq \theta \leq 2pi$

Step 4: evaluate

$\int_0^(2pi) \int_0^2 r^2 * r dr dtheta = \int_0^(2pi) \int_0^2 r^3 dr dtheta$.

The inner integral is `4`, so the total is

$8pi$.

Example 12. Conservative field shortcut

Let $F(x, y) = \langle 2x, 2y \rangle$. Find the work done from `(1,1)` to `(3,2)`.

Step 1: find a potential

A potential function is

$f(x, y) = x^2 + y^2$.

Step 2: evaluate at endpoints

$f(3,2) - f(1,1) = (9 + 4) - (1 + 1) = 11$.

Conclusion

The work is `11`.

Why this example matters

When a field is conservative, endpoint evaluation replaces a full path computation.

Example 13. Flux interpretation on a simple surface

Suppose $F(x, y, z) = \langle 0, 0, 7 \rangle$. Describe the flux through a horizontal upward-facing square.

Step 1: compare field direction with normal direction

The field points straight upward, and the square's upward normal also points upward.

Step 2: interpret

The field passes directly through the surface, so the flux is positive and as large as possible for that field strength and area.

Contrast case

If the same field met a vertical wall, the flux would be `0` because the field would run parallel to the surface rather than across it.

Example 14. Connecting the major integral theorems

Explain how the Fundamental Theorem of Calculus and the Divergence Theorem are philosophically related.

Step 1: identify the local quantity

In the Fundamental Theorem, the local quantity is the derivative. In the Divergence Theorem, the local quantity is divergence.

Step 2: identify the global quantity

In the Fundamental Theorem, the global quantity is endpoint change. In the Divergence Theorem, the global quantity is total outward flux across the boundary.

Step 3: state the shared pattern

Both theorems convert accumulated local differential behavior in the interior into a boundary quantity. The geometry changes, but the logic is the same.

Why this example matters

A good calculus text should not leave the final theorems looking unrelated. They are parallel versions of one central idea.

Example 15. Using the integral test to control a tail

Estimate the error made by replacing

$$\sum_{(} n = 1)^{i}nfinity1/n^{2}$$

with the partial sum through $n = 10$.

Step 1: identify the remainder

The remainder is

$R_{1}0 = \sum_{(} n = 11)^{i}nfinity1/n^{2}$.

Step 2: use the integral bound

Because $1/x^{2}$ is positive and decreasing,

$\int_{1} 1^{i}nfinity1/x^{2}dx \leq R_{1}0 \leq \int_{1} 0^{i}nfinity1/x^{2}dx.$

Step 3: evaluate the bounds

$\int_{a}^{i} nfinity1/x^{2}dx = [-1/x]_{a}^{i}nfinity = 1/a.$

So

$1/11 \leq R_{1}0 \leq 1/10.$

Interpretation

The tail after the tenth term is less than `0.1`, and in fact squeezed between about `0.091` and `0.1`.

Why this example matters

Convergence is one question. Error control is another. Good series work answers both.

Example 16. Radius and interval of convergence

Find the radius and interval of convergence of

$\sum_(n = 1)^infinity n((x - 2)^n)/(3^n)$.

Step 1: apply the ratio test

Let

$a_n = n((x - 2)^n)/(3^n)$.

Then

$|a_(n + 1)/a_n| = ((n + 1)/n)|x - 2|/3$.

As $n \to infinity$, this approaches

`|x - 2|/3`.

Step 2: determine the radius

The ratio test gives convergence when

$|x - 2|/3 < 1$,

so

$|x - 2| < 3$.

Therefore the radius of convergence is `3`.

Step 3: test the endpoints

- At $x = 5$, the series becomes $sum n$, which diverges.
- At $x = -1$, the series becomes $sum n(-1)^n$, whose terms do not approach `0`, so it diverges.

Conclusion

The interval of convergence is

`(-1, 5)`.

Common mistake

The ratio test gives the open interval first. Endpoints must still be checked separately.

Example 17. A damped oscillator

Solve

$$y + 2y + 10y = 0$$

with $y(0) = 1$ and $y'(0) = 0$.

Step 1: solve the characteristic equation

$r^2 + 2r + 10 = 0.$

Using the quadratic formula,

$r = -1 + -3i.$

Step 2: write the general solution

For complex roots $\alpha + -\beta i$, the solution has the form

$y = e^{(}\alpha t)(C_1 \cos \beta t + C_2 \sin \beta t).$

So here

$y = e^{(} - t)(C_1 \cos 3t + C_2 \sin 3t).$

Step 3: use the initial conditions

From $y(0) = 1$, we get $C_1 = 1$.

Differentiate:

$y' = e^{(} - t)(-C_1 \cos 3t - C_2 \sin 3t - 3C_1 \sin 3t + 3C_2 \cos 3t).$

At $t = 0$,

$y'(0) = -C_1 + 3C_2 = 0.$

Since $C_1 = 1$, we get $C_2 = 1/3$.

Conclusion

$y = e^{(-t)}(\cos 3t + (1/3)\sin 3t).$

Interpretation

The oscillation persists, but the factor $e^{(-t)}$ causes the amplitude to decay over time.

Example 18. Projection onto a line

Project $a = \langle 3, 4, -1 \rangle$ onto $b = \langle 1, 2, 2 \rangle$.

Step 1: compute the dot product and squared norm

$a \cdot b = 3 + 8 - 2 = 9.$

$|b|^2 = 1 + 4 + 4 = 9.$

Step 2: apply the projection formula

$proj_b a = ((a \cdot b)/|b|^2) b.$

So

$proj_b a = (9/9)\langle 1, 2, 2 \rangle = \langle 1, 2, 2 \rangle.$

Interpretation

The component of `a` in the direction of `b` is exactly the vector `b`.

Why this example matters

Projection is the geometric content hiding inside many formulas involving the dot product.

Example 19. Tangent plane and linear estimate

Let $f(x, y) = x^2 + y^2$. Find the tangent plane at `(1, 2)` and use it to estimate $f(1.1, 1.9)$.

Step 1: compute the point on the surface

$f(1,2) = 5.$

So the surface point is `(1, 2, 5)`.

Step 2: compute the partial derivatives

- $f_x = 2x,$
- $f_y = 2y.$

At `(1, 2)`,

$f_x = 2, f_y = 4.$

Step 3: write the tangent-plane formula

$z = f(a,b) + f_x(a,b)(x - a) + f_y(a,b)(y - b).$

So

$z = 5 + 2(x - 1) + 4(y - 2).$

Step 4: estimate the nearby value

At `(1.1, 1.9)`,

$z \approx 5 + 2(0.1) + 4(-0.1) = 4.8.$

Check

The exact value is

$(1.1)^2 + (1.9)^2 = 1.21 + 3.61 = 4.82,$

which is close to the linear estimate.

Example 20. Lagrange multipliers on a circle

Find the maximum and minimum values of $f(x,y) = xy$ subject to the constraint

$x^2 + y^2 = 1.$

Step 1: compute gradients

$\nabla f = \langle y, x \rangle$,

$\nabla g = \langle 2x, 2y \rangle$ for $g(x, y) = x^2 + y^2$.

Step 2: set $\nabla f = \lambda \nabla g$

This gives

- $y = 2\lambda x$,
- $x = 2\lambda y$.

Step 3: solve the system

If `x` and `y` are both nonzero, then combining the equations shows $x = +- y$.

Use the constraint:

- if $x = y$, then $2x^2 = 1$, so $x = +- 1/\sqrt{2}$,
- if $x = -y$, then $2x^2 = 1$, so $x = +- 1/\sqrt{2}$.

Step 4: evaluate the objective

- at $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$, $xy = 1/2$,
- at $(1/\sqrt{2}, -1/\sqrt{2})$ and $(-1/\sqrt{2}, 1/\sqrt{2})$, $xy = -1/2$.

Conclusion

Maximum value: $1/2$.
Minimum value: $-1/2$.

Why this example matters

Lagrange multipliers organize constrained optimization by forcing the objective gradient to line up with the constraint gradient.

Example 21. A triple integral in cylindrical coordinates

Find the volume of the solid under

$$z = 4 - x^2 - y^2$$

and above the plane $z = 0$.

Step 1: identify the projection in the `xy`-plane

The surface meets $z = 0$ when

$x^2 + y^2 = 4.$

So the base is the disk $r \le 2$.

Step 2: rewrite in cylindrical coordinates

Because $x^2 + y^2 = r^2$, the top surface is

$z = 4 - r^2.$

The volume integral is

$\int_0^(2pi) \int_0^2 \int_0^(4 - r^2) r\,dz\,dr\,dtheta.$

Step 3: integrate

The `z`-integral gives

$\int_0^(2pi) \int_0^2 r(4 - r^2)\,dr\,dtheta.$

Now

$\int_0^2 (4r - r^3)dr = [2r^2 - r^4/4]_0^2 = 8 - 4 = 4.$

Multiplying by the θ-range gives

$8pi.$

Why this example matters

When a solid is rotationally symmetric, cylindrical coordinates usually turn the geometry into cleaner bounds.

Example 22. A line integral along a parametrized path

Compute the work integral

$$\int_C F \cdot dr$$

for $F(x,y) = \langle y, x \rangle$ along the line segment from `(0, 0)` to `(1, 2)`.

Step 1: parametrize the curve

A natural parametrization is

$r(t) = \langle t, 2t \rangle, 0 \leq t \leq 1.$

Then

$r'(t) = \langle 1, 2 \rangle.$

Step 2: evaluate the field on the curve

$F(r(t)) = \langle 2t, t \rangle.$

Step 3: take the dot product

$F(r(t)).r'(t) = \langle 2t, t \rangle. \langle 1, 2 \rangle = 4t.$

Step 4: integrate

$\int_0^1 4t dt = 2.$

Conclusion

The work done along the path is `2`.

Example 23. Green's Theorem on a unit circle

Use Green's Theorem to evaluate the circulation of

$$F(x,y) = \langle -y, x \rangle$$

around the unit circle oriented counterclockwise.

Step 1: identify `P` and `Q`

$P = -y, Q = x.$

Step 2: compute the scalar curl

$dQ/dx - dP/dy = 1 - (-1) = 2.$

Step 3: integrate over the enclosed region

The enclosed region is the unit disk, whose area is π.

So Green's Theorem gives

$\int_C F \cdot dr = \iint_R 2 dA = 2pi.$

Interpretation

The field circles around the origin, and the theorem converts that circulation into an area computation.

Example 24. Divergence Theorem on a sphere

Find the outward flux of

$$F(x, y, z) = \langle x, y, z \rangle$$

across the sphere of radius `2`.

Step 1: compute the divergence

$\text{div}\, F = 1 + 1 + 1 = 3.$

Step 2: compute the volume of the enclosed solid

The volume of a sphere of radius `2` is

$(4/3)\pi(2^3) = 32pi/3.$

Step 3: apply the Divergence Theorem

Flux equals

$\iiint_E \text{div}\, F\, dV = \iiint_E 3 dV = 3 * 32pi/3 = 32pi.$

Why this example matters

The theorem replaces a curved surface computation with a simple volume calculation.

Example 25. Stokes' Theorem on a flat disk

Use Stokes' Theorem to compute

$$\int_C F \cdot dr$$

for

$$F(x, y, z) = \langle -y, x, 0 \rangle$$

where `C` is the unit circle in the plane $z = 0$, oriented counterclockwise as viewed from above.

Step 1: compute the curl

$\mathrm{curl}\, F = \langle 0, 0, 2 \rangle$.

Step 2: choose the surface

Use the flat unit disk with upward normal $n = \langle 0, 0, 1 \rangle$.

Step 3: compute the flux of the curl

$\mathrm{curl}\, F \cdot n = 2$.

So

$\iint_S \mathrm{curl}\, F \cdot n\, dS = \iint_S 2 dS = 2pi$.

Conclusion

By Stokes' Theorem,

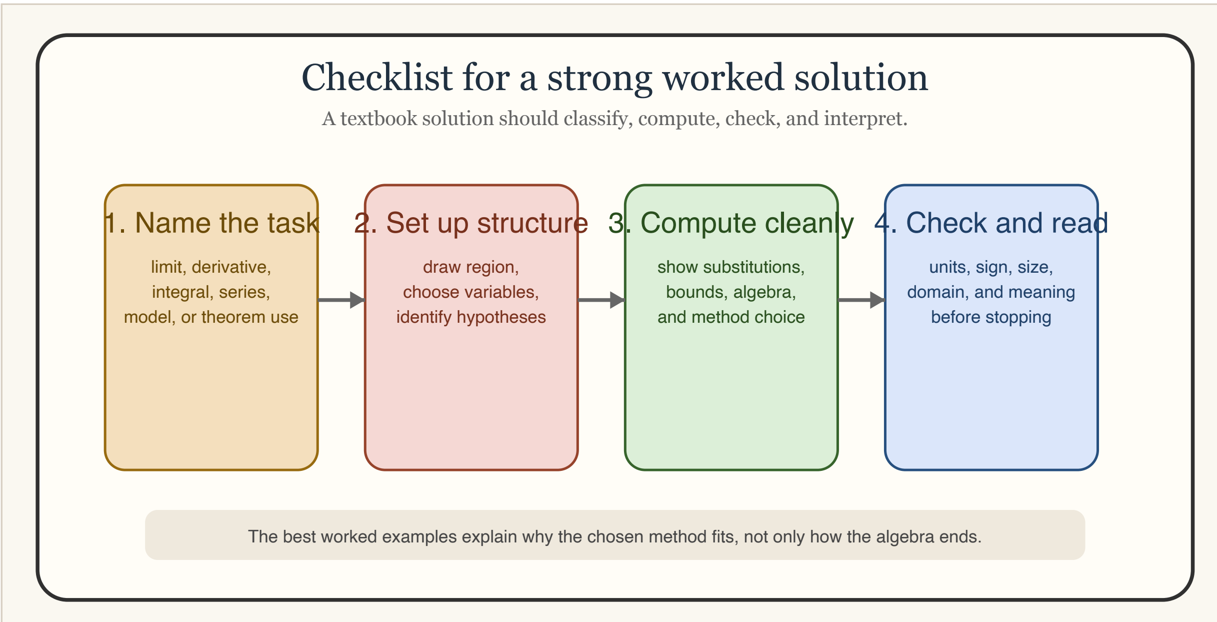
$\int_C F \cdot dr = 2pi$.

Why this example matters

Stokes' Theorem is the three-dimensional analogue of Green's Theorem: circulation around a boundary equals accumulated local turning across the spanning surface.

Appendix V. Worked Solution Atlas

This appendix adds one representative worked solution for each chapter. It is not meant to replace full homework, but to model the level of setup, interpretation, and checking that students should aim for in classroom work.



How to use this appendix

- Read the problem statement first and decide on a method before reading the solution.
- Cover the final line and try to finish the work independently.
- Pay attention to the setup and interpretation sentences, not only the algebra.

Chapter 1. Functions and change

`Problem`

A reservoir rises from `240` gallons to `312` gallons in `6` hours. Find the average rate of change and interpret it.

`Solution`

Average rate of change is output change divided by input change:

$(312 - 240)/6 = 72/6 = 12.$

So the average rate is `12` gallons per hour. The interpretation matters: this does not say the inflow was exactly `12` gallons per hour at every instant. It says that over the whole six-hour interval, the total change was the same as if the reservoir had risen steadily at `12` gallons per hour.

Chapter 2. Limits and continuity

`Problem`

Compute $\lim_{(}x \to 2)(x^2 - 4)/(x - 2)$.

`Solution`

Direct substitution gives $0/0$, so the algebra must be simplified before evaluating. Factor the numerator:

$x^2 - 4 = (x - 2)(x + 2).$

For $x \neq 2$, the expression becomes `x + 2`. The nearby behavior is therefore the same as the behavior of `x + 2`, even though the original formula is undefined at $x = 2$. Taking the limit now gives `4`. This is a standard removable-discontinuity pattern: the function may have a hole, but the limit still exists.

Chapter 3. Derivatives

`Problem`

Use the limit definition to find the derivative of $f(x) = x^2$ at a general point `x`.

`Solution`

Start from the definition:

$f'(x) = \lim_{(}h \to 0)((x + h)^2 - x^2)/h.$

Expand the numerator:

$(x + h)^2 - x^2 = 2xh + h^2.$

Then

$f'(x) = \lim_{(}h \to 0)(2xh + h^2)/h = \lim_{(}h \to 0)(2x + h) = 2x.$

The algebraic cancellation is not the main idea. The main idea is that the derivative comes from the limiting behavior of average slopes over shorter and shorter intervals.

Chapter 4. Working with derivatives

`Problem`

Differentiate $y = (3x^2 + 1)^5$.

`Solution`

This is a composition, so the chain rule is the correct tool. Let the outer function be u^5 and the inner function be $u = 3x^2 + 1$. Differentiate in two stages:

- derivative of the outer form: $5u^4$,
- derivative of the inner form: `6x`.

Multiply them:

$y' = 5(3x^2 + 1)^4(6x) = 30x(3x^2 + 1)^4$.

The common mistake is to differentiate only the outer power and forget the inner derivative.

Chapter 5. What derivatives tell us

`Problem`

Find and classify the critical points of $f(x) = x^3 - 3x$.

`Solution`

Differentiate:

$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$.

Critical points occur where $f' = 0$, so $x = -1$ and $x = 1$. Use a sign chart for `f'`:

- positive on `(-infinity, -1)`,
- negative on `(-1, 1)`,
- positive on `(1, infinity)`.

So the function rises then falls at $x = -1$, giving a local maximum there, and falls then rises at $x = 1$, giving a local minimum there. The classification comes from sign change, not from the derivative being zero by itself.

Chapter 6. The integral as accumulation

`Problem`

Interpret $\int_0^3 (4 - x)dx$ and evaluate it.

`Solution`

The integral represents net accumulation of the quantity `4 - x` from $x = 0$ to $x = 3$. Since the function stays positive on that interval, it is also the ordinary geometric area under the line. Compute:

$\int_0^3 (4 - x)dx = [4x - x^2/2]_0^3 = 12 - 9/2 = 15/2.$

So the accumulated total is $15/2$. If a context were attached, the units would be "units of quantity times units of input."

Chapter 7. The Fundamental Theorem

`Problem`

Evaluate $\int_0^1 2x(x^2 + 1)^3 dx$.

`Solution`

The factor `2x` signals substitution because it is the derivative of $x^2 + 1$. Let $u = x^2 + 1$, so $du = 2x dx$. Change the bounds:

- when $x = 0, u = 1$,
- when $x = 1, u = 2$.

Then

$\int_0^1 2x(x^2 + 1)^3 dx = \int_1^2 u^3 du = [u^4/4]_1^2 = (16 - 1)/4 = 15/4.$

The structural skill is seeing the inside derivative, not memorizing the answer.

Chapter 8. More integration tools

`Problem`

Evaluate $int \times e^x dx$.

`Solution`

This is a standard integration-by-parts shape because an algebraic factor becomes simpler when differentiated, while e^x stays manageable when integrated. Choose:

- $u = x$, so $du = dx$,
- $dv = e^x dx$, so $v = e^x$.

Then

$int \times e^x dx = xe^x - inte^x dx = xe^x - e^x + C = e^x(x - 1) + C.$

The method choice is the real lesson here.

Chapter 9. Applications of integration

`Problem`

Set up and evaluate the area between $y = x$ and $y = x^2$ on `[0, 1]`.

`Solution`

On `[0, 1]`, the line $y = x$ lies above the parabola $y = x^2$, so the area is

$\int_0^1 (x - x^2) dx.$

Now integrate:

$[x^2/2 - x^3/3]_0^1 = 1/2 - 1/3 = 1/6.$

The main setup choice is top minus bottom. If that order were reversed, the integral would give negative signed area instead of the positive enclosed area.

Chapter 10. Sequences and series

`Problem`

Decide whether $\Sigma_{(} n = 1)^{i} nfinity 1/n^2$ converges.

`Solution`

This is a positive-term series. A standard comparison or the `p`-series test applies. Since the exponent is $p = 2$, and $p > 1$, the series converges. If the student wants a continuous justification, the integral

$$\int_1^{i} nfinity 1/x^2 dx$$

also converges, which agrees with the integral test. The important distinction is that the series is about the partial sums, not only about the individual terms $1/n^2$.

Chapter 11. Differential equations and models

`Problem`

Solve $dy/dx = ky$ with $y(0) = 5$.

`Solution`

Separate variables:

$(1/y)dy = kdx.$

Integrate:

$\ln|y| = kx + C.$

Exponentiate:

$y = Ce^{kx}.$

Use the initial condition $y(0) = 5$ to get $C = 5$. Therefore

$y = 5e^{kx}.$

The equation says the rate is proportional to the current amount, so an exponential answer is not only algebraically correct but structurally natural.

Chapter 12. Vectors and space

`Problem`

Find the tangent line to $r(t) = \langle t, t^2, t^3 \rangle$ at $t = 1$.

`Solution`

First compute the point:

$r(1) = \langle 1, 1, 1 \rangle.$

Then differentiate:

$r'(t) = \langle 1, 2t, 3t^2 \rangle,$

so

$r'(1) = \langle 1, 2, 3 \rangle.$

The tangent line uses the point and the tangent vector:

$L(s) = \langle 1, 1, 1 \rangle + s\langle 1, 2, 3 \rangle.$

This mirrors single-variable tangent-line logic, except that both the point and the direction now live in three dimensions.

Chapter 13. Multivariable functions

`Problem`

Find the gradient of $f(x, y) = x^2 + 3y^2$ and evaluate it at `(1, -2)`.

`Solution`

Compute partial derivatives:

- $f_x = 2x,$
- $f_y = 6y.$

So

$\nabla f(x, y) = \langle 2x, 6y \rangle.$

At `(1, -2)`,

$\nabla f(1, -2) = \langle 2, -12 \rangle.$

The gradient is more than a vector of partials. It points in the direction of steepest increase, so the negative `y`-component says the function decreases rapidly when moving upward in `y` from that point.

Chapter 14. Multiple integration

`Problem`

Set up the double integral of $f(x, y) = x + y$ over the rectangle $0 \le x \le 2, 0 \le y \le 1.$

`Solution`

Because the region is rectangular, the iterated integral is immediate:

$\int_0^2 \int_0^1 (x + y) dy dx.$

You could also reverse the order. The main step is not computing but reading the region correctly. Double integrals accumulate a quantity over a two-dimensional set, and iterated integrals simply organize that accumulation in stages.

Chapter 15. Vector calculus

`Problem`

Explain when a vector field is conservative.

`Solution`

A vector field is conservative when it is the gradient of some scalar potential function. In that case, line integrals depend only on endpoints, not on the path taken. For classroom use, there are several equivalent viewpoints:

- there exists a potential,
- work is path independent,
- the integral around a closed loop is zero on a suitable domain.

The definition is not only symbolic. It is about whether local directional data can be organized into a global scalar landscape.

Chapter 16. Parametric curves, polar coordinates, and curvature

`Problem`

Find the slope of $x = t^2 + 1$, $y = t^3 - t$ at $t = 1$.

`Solution`

Differentiate both coordinates:

- $dx/dt = 2t$,
- $dy/dt = 3t^2 - 1$.

Then

$dy/dx = (3t^2 - 1)/(2t).$

At $t = 1$, the slope is $(3 - 1)/2 = 1$. The point is `(2, 0)`, so the tangent line there is $y = x - 2$. The key habit is to compute the point and the slope separately before writing the line.

Chapter 17. Second-order differential equations and oscillation

`Problem`

Solve $y'' + 4y = 0$ with $y(0) = 3$ and $y'(0) = -2$.

`Solution`

The characteristic equation is

$r^2 + 4 = 0,$

so the roots are `+- 2i`. Therefore

$y = C_1 \cos 2t + C_2 \sin 2t.$

Use $y(0) = 3$ to get $C_1 = 3$. Differentiate:

$y' = -2C_1 \sin 2t + 2C_2 \cos 2t.$

Then $y'(0) = 2C_2 = -2$, so $C_2 = -1$. The solution is

$y = 3\cos 2t - \sin 2t.$

This describes undamped oscillation with angular frequency `2`.

Chapter 18. Improper integrals and long-tail behavior

`Problem`

Determine whether $\int_1^i nfinity1/x^2dx$ converges and interpret the result.

`Solution`

Because the interval is unbounded, rewrite the integral as a limit:

$\int_1^i nfinity1/x^2dx = \lim_(b \to infinity) \int_1^b x^(-2)dx.$

Now compute:

$\int_1^b x^{(}-2)dx = [-1/x]_1^b = 1 - 1/b.$

As $b \to$ infinity, the term `1/b` tends to `0`, so the improper integral converges to `1`. The interpretation is important: the region stretches infinitely far to the right, but the total accumulated area settles to a finite value because the tail becomes thin fast enough.

Extended clinic 1. Rationalizing a radical limit

`Problem`

Compute

$\lim_{(} x \to 0)(\sqrt{1+x} - 1)/x.$

`Solution`

Direct substitution gives $0/0$, so the expression needs algebraic repair before the limit can be taken. Multiply by the conjugate:

$(\sqrt{1+x} - 1)/x * (\sqrt{1+x} + 1)/(\sqrt{1+x} + 1).$

The numerator becomes

$(1+x) - 1 = x.$

So for $x \neq 0$,

$(\sqrt{1+x} - 1)/x = 1/(\sqrt{1+x} + 1).$

Now the limit is immediate:

$\lim_{(} x \to 0) 1/(\sqrt{1+x} + 1) = 1/2.$

The main lesson is structural. Rationalizing is not a trick for radicals in general. It is specifically a way to expose cancellation when a difference of square roots hides the decisive factor.

Extended clinic 2. Derivative with units and interpretation

`Problem`

The temperature `T` of a reactor vessel after `t` minutes is modeled by

$T(t) = 80 + 15e^{(} - 0.2t).$

Find `T'(t)` and interpret `T'(5)`.

`Solution`

Differentiate carefully:

$T'(t) = 15(-0.2)e^{(}-0.2t) = -3e^{(}-0.2t).$

Now evaluate at $t = 5$:

$T'(5) = -3e^{(}-1) \approx -1.10.$

The units matter. Since temperature is measured in degrees and time in minutes, `T'(5)` is measured in degrees per minute. The negative sign says the temperature is decreasing, and the size says that at $t = 5$ minutes the vessel is cooling at about `1.10` degrees per minute.

Students often stop after the symbolic derivative. A complete worked solution goes one line further and says what the number means in the original system.

Extended clinic 3. Optimization with a river boundary

`Problem`

A farmer has `600` meters of fencing and wants to build a rectangular pen along a straight river, so no fence is needed on the river side. What dimensions maximize the area?

`Solution`

Let `x` be the width perpendicular to the river and `y` the length parallel to the river. Only three sides require fencing, so the constraint is

$2x + y = 600.$

Thus

$y = 600 - 2x.$

Area is

$A = xy = x(600 - 2x) = 600x - 2x^2.$

Differentiate:

$A'(x) = 600 - 4x.$

Set $A'(x) = 0$:

$600 - 4x = 0$, so $x = 150$.

Then

$y = 600 - 300 = 300$.

Because the quadratic opens downward, this critical point gives a maximum. The optimal pen is `150` meters by `300` meters. The geometry matters here: without translating the fencing constraint into one variable, there is no calculus problem yet.

Extended clinic 4. Average value and the Mean Value Theorem for integrals

`Problem`

Find the average value of $f(x) = 3x^2 + 2$ on `[0, 2]`, and identify a value `c` guaranteed by the Mean Value Theorem for integrals.

`Solution`

The average value is

$f_avg = (1/(2 - 0)) \int_0^2 (3x^2 + 2)dx = (1/2)[x^3 + 2x]_0^2$.

So

$f_avg = (1/2)(8 + 4) = 6$.

Now solve $f(c) = 6$:

$3c^2 + 2 = 6$,

so

$3c^2 = 4$, hence $c^2 = 4/3$ and

$c = 2/\sqrt{3}$.

This lies in `[0, 2]`, so it is a valid choice. A good interpretation sentence is: there is at least one point where the instantaneous height of the graph matches the constant height that would produce the same accumulated area over the interval.

Extended clinic 5. Choosing a series test with justification

`Problem`

Determine whether

$$\sum_{n=2}^{\infty} 1/(n \ln n)$$

converges.

`Solution`

This is a positive-term series, so a comparison-style argument or the integral test is appropriate. The integral test is clean because the function

$$f(x) = 1/(x \ln x)$$

is positive and decreasing for $x > 1$.

Compute

$\int_2^b 1/(x \ln x)dx.$

Use $u = \ln x$, so $du = dx/x$. Then

$\int_2^b 1/(x \ln x)dx = \int_{\ln 2}^{\ln b} 1/u\,du = [\ln u]_{\ln 2}^{\ln b}$.

As $b \to \infty$, the quantity $\ln(\ln b)$ also tends to infinity. Therefore the integral diverges, and so does the series.

The important habit is not just naming a test but explaining why the test matches the structure of the terms.

Extended clinic 6. Taylor approximation with an error bound

`Problem`

Use the quadratic Maclaurin polynomial for $\sin x$ to approximate $\sin(0.4)$ and give a bound on the error from the next nonzero term.

`Solution`

The Maclaurin series for $\sin x$ begins

$\sin x = x - x^3/3! + x^5/5! - \ldots.$

Keeping terms through degree `3` gives

$P_3(x) = x - x^3/6.$

So

$P_3(0.4) = 0.4 - 0.4^3/6 = 0.4 - 0.064/6 \approx 0.38933.$

The next nonzero omitted term has magnitude

$0.4^5/5! = 0.01024/120 \approx 0.0000853.$

So the approximation error is at most about $8.53 \times 10^{(-5)}$ in magnitude. This is a model of a good textbook solution: compute the approximation, then say why the reader should trust it.

Extended clinic 7. Solving and interpreting a logistic model

`Problem`

Solve

$$dP/dt = 0.5P(1 - P/1000)$$

with $P(0) = 200$, and identify the long-term behavior.

`Solution`

The carrying capacity is `1000`, and the initial value is positive and below capacity, so we expect growth toward `1000`.

Separate variables:

$dP/[P(1 - P/1000)] = 0.5dt.$

The standard logistic solution form is

$P(t) = K/(1 + Ae^{(-rt)}).$

Here $K = 1000$ and $r = 0.5$, so

$P(t) = 1000/(1 + Ae^{(-0.5t)}).$

Use $P(0) = 200$:

$200 = 1000/(1 + A)$,

so $1 + A = 5$ and $A = 4$.

Therefore

$P(t) = 1000/(1 + 4e^{(} - 0.5t))$.

As $t \to infinity$, the exponential term goes to `0`, so $P(t) \to 1000$. The long-term interpretation is that the population approaches the carrying capacity and levels off rather than growing exponentially forever.

Extended clinic 8. A plane through a point with a given normal

`Problem`

Find the equation of the plane through `(1, -2, 4)` with normal vector $\langle 2, -1, 3 \rangle$.

`Solution`

The point-normal form of a plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

when $\langle a, b, c \rangle$ is a normal vector. Substituting the given data gives

$2(x - 1) - (y + 2) + 3(z - 4) = 0$.

Expand:

$2x - 2 - y - 2 + 3z - 12 = 0$.

So

$2x - y + 3z - 16 = 0$.

This is a good place to check. Plugging the point back in gives

$2(1) - (-2) + 3(4) - 16 = 0$,

so the equation is consistent.

Extended clinic 9. Constrained optimization with Lagrange multipliers

`Problem`

Find the maximum and minimum values of $f(x, y) = x + y$ on the ellipse

$x^2 + 4y^2 = 4.$

`Solution`

Let $g(x, y) = x^2 + 4y^2$. Then

$\nabla f = \langle 1, 1 \rangle,$

$\nabla g = \langle 2x, 8y \rangle.$

Set

$\nabla f = \lambda \nabla g,$

so

- $1 = 2\lambda x,$
- $1 = 8\lambda y.$

From these, $2x = 8y$, hence $x = 4y$.

Use the constraint:

$(4y)^2 + 4y^2 = 4,$

so

$20y^2 = 4$, giving $y = +- 1/\sqrt{5}.$

Then $x = +- 4/\sqrt{5}$ with the same sign.

Evaluate `f`:

- maximum value $= 5/\sqrt{5} = \sqrt{5},$
- minimum value $= -\sqrt{5}.$

The method works because at an extreme point on a smooth constraint curve, the objective's steepest-change direction cannot point across the constraint; it must line up with the constraint normal.

Extended clinic 10. Reversing order and evaluating

`Problem`

Reverse the order of integration and evaluate

$\int_0^1 \int_y^1 (x + y)\,dx\,dy.$

`Solution`

The original region is given by

- $0 \leq y \leq 1,$
- $y \leq x \leq 1.$

This is the triangle below the line $y = x$ in the unit square. If we sweep vertically instead, then

- $0 \leq x \leq 1,$
- $0 \leq y \leq x.$

So the reversed integral is

$\int_0^1 \int_0^x (x + y)\,dy\,dx.$

Now compute the inner integral:

$\int_0^x (x + y)\,dy = [xy + y^2/2]_0^x = x^2 + x^2/2 = 3x^2/2.$

Then

$\int_0^1 3x^2/2\,dx = [x^3/2]_0^1 = 1/2.$

The picture is the decisive step. Without the region sketch, reversing order is guesswork.

Extended clinic 11. Flux from the Divergence Theorem

`Problem`

Find the outward flux of $F(x, y, z) = \langle x^2, y^2, z^2 \rangle$ across the boundary of the cube $0 \leq x, y, z \leq 1.$

`Solution`

The Divergence Theorem turns a surface-flux problem into a triple integral:

$\iint_S F \cdot n dS = \iiint_E \operatorname{div} F dV.$

Compute the divergence:

$\operatorname{div} F = 2x + 2y + 2z.$

So the flux is

$\int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z) dz dy dx.$

By symmetry and separability,

$$= 2\int_0^1 \times dx + 2\int_0^1 y dy + 2\int_0^1 z dz$$

over unit measure in the other variables, giving

$1 + 1 + 1 = 3.$

If the reader wants a direct check, each face has simple polynomial flux, but the theorem is the efficient route because the field and solid are both simple enough for divergence to dominate the calculation.

Extended clinic 12. Improper integral by comparison

`Problem`

Determine whether

$$\int_1^i nfinity 1/(x^2 + 4x + 3) dx$$

converges.

`Solution`

For large `x`, the denominator behaves like x^2, so comparison with $1/x^2$ is natural. For $x \geq 1$
,

$x^2 + 4x + 3 \geq x^2.$

Therefore

$0 \leq 1/(x^2 + 4x + 3) \leq 1/x^2.$

Since

$$\int_1^{i} nfinity1/x^2dx$$

converges, the comparison test shows that

$$\int_1^{i} nfinity1/(x^2 + 4x + 3)dx$$

also converges.

This is a good example of a full solution that stops at the right moment. If the question is only convergence, a clean comparison argument is enough. An explicit antiderivative is unnecessary.

Answers and Hints

This back-of-book key is selective rather than exhaustive. It is designed to support classroom use without replacing the student's own work. Short answers are paired with occasional worked hints so that the appendix teaches strategy as well as outcome. Exercise numbers run continuously within each chapter, even when the chapter exercise section is split into `Warm-up`, `Core skill`, `Interpretation`, `Challenge`, and `Modeling`.

How to use this key

- Check a short answer only after you have completed a full solution.
- Use the worked hints to diagnose method errors, not only arithmetic errors.
- If your numerical answer differs, compare your setup before comparing your computation.
- For fuller model solutions, use the `Worked Solution Atlas` alongside this appendix.

Chapter 1

1. `-24` liters per hour.
2. `30` kilometers per hour.
3. $14/5 = 2.8$.
4. `12` dollars per unit. The linear model has the same average rate on every interval.
5. 4.5pi.

`Hint for 5`

Keep the geometry separate from the arithmetic. First identify the radius from the context, then apply the circle or sector formula carefully before inserting π.

Chapter 2

1. The outputs get close to `9` when the inputs get close to `4`.
2. `7`.
3. The two-sided limit exists and equals `5`.
4. `9`.
5. `-3`.

6. 13/6.
7. `4`.
8. Left-hand limit `1`, right-hand limit `4`, two-sided limit does not exist.
9. Assign `6`.

`Worked hint for 8`

Do not start by asking for the function value at $x = 3$. Read the rule on each side of `3` separately. The left-hand expression gives one constant value and the right-hand expression gives another. Because those one-sided values differ, no two-sided limit can exist.

Chapter 3

1. The slope of the tangent line.
2. Instantaneous velocity.
3. The function is increasing locally.
4. `4`.
5. `5`.
6. The left-hand slopes approach `−1` and the right-hand slopes approach `1`.

`Worked hint for 4`

Start with the difference quotient from the definition. Simplify fully before taking the limit. Many errors in Chapter 3 come from substituting too early into an unsimplified quotient.

Chapter 4

1. $6x^5$.
2. $12x^3 - 7$.
3. The inside derivative, which is `5`.
4. $14(2x - 3)^6$.
5. $4e^{4x}$.
6. $2x/(x^2 + 9)$.

`Worked hint for 6`

Recognize the structure before differentiating. This is not just a power of `x`; it is a power of an inner linear expression. Apply the power rule to the outer form and then multiply by the derivative of the inside.

Chapter 5

1. The function increases there.
2. A point where $f'(x) = 0$ or `f'(x)` does not exist.
3. Concave up.
4. Increasing on `(-infinity, -1)` and `(1, infinity)`; decreasing on `(-1, 1)`.
5. No; it is a flat point but not an extremum.
6. The maximizing production level is at the vertex, $x = 60$.

`Worked hint for 4`

Factor the derivative first, then test the sign of each interval rather than plugging many values into the original function. The derivative sign chart is the real tool here.

`Worked hint for 15`

When an optimization answer is given as an `x`-value, finish the job by translating it back into the original context. A production level, width, time, or cost parameter must be named in words and units.

Chapter 6

1. The accumulated total or net change over the interval.
2. It contributes negatively to displacement.
3. `7`.
4. `10`.
5. `6`.
6. `8`.

`Worked hint for 2`

Negative integrand values do not mean the integral is "bad." They mean the local contribution points in the negative direction of the modeled quantity.

Chapter 7

1. $3x^2$.
2. Because all antiderivatives differ by a constant.
3. $\int_a^b f(x)dx = F(b) - F(a)$ when $F' = f$.
4. `8`.
5. $x^2 + 4$.
6. `e - 1`.

`Worked hint for 8`

Use substitution with $u = x^2$. In a definite integral, remember to change the bounds as well as the variable.

Chapter 8

1. $intudv = uv - intvdu$.
2. Because it is the derivative of $\tan x$.
3. To estimate definite integrals when exact symbolic integration is unavailable or unnecessary.
4. $e^x(x - 1) + C$.
5. $\tan x + C$.
6. $(1/2)\ln(x^2 + 1) + C$.

`Worked hint for 4`

Choose the algebraic factor as `u` so it simplifies when differentiated. The exponential factor stays manageable when integrated. That is the standard shape for a clean integration-by-parts problem.

`Worked hint for 7`

This is not really a partial-fractions problem. It is a substitution problem because the denominator's derivative appears in the numerator up to a constant factor.

Chapter 9

1. $\int_a^b(\text{top} - \text{bottom})dx$.
2. The cross-sectional area at position `x`.
3. Mass per unit length.
4. $1/6$.
5. $\pi/3$.
6. `8`.
7. `18`.

`Worked hint for 5`

Identify the radius of the disk before writing the volume integral. The integrand should reflect area of a cross section, not just the original function itself.

Chapter 10

1. The terms approach a single limit.
2. A sum $a_1 + \ldots + a_N$.
3. When $|r| < 1$.
4. Yes, it converges to `0`.
5. No, it oscillates and diverges.
6. `4`.
7. It converges.
8. It diverges.
9. Converges by comparison with $1/n^2$.
10. Converges by comparison with $1/n^2$.
11. Diverges.
12. Converges absolutely for every fixed real `x`.
13. Conditionally convergent.
14. Absolutely convergent.
15. The three-term partial sum is $13/15$, and the error is at most $1/7$.
16. $1 + x + x^2/2 + x^3/6$.
17. $x - x^3/6 + x^5/120 - x^7/5040$.
18. $0.1 - 0.1^3/6 = 0.0998333\ldots$.

`Worked hint for 6`

For a geometric series, isolate the first term and the ratio first. Do not expand more terms than necessary once the structure is visible.

`Worked hint for 21`

When factorial growth appears in the numerator, the ratio test is usually the first method to try. If the ratio eventually becomes larger than `1`, the terms are not shrinking fast enough for convergence.

Chapter 11

1. A picture of the local slopes determined by the differential equation.
2. It can be rewritten so `y` terms go with dy and `x` terms with dx.
3. The equilibrium population level.
4. $y = (3/2)x^2 + C$.
5. $y = Ce^{4x}$.
6. $P(t) = 50e^{0.1t}$.
7. $y = 0$ and $y = 4$.
8. Negative, positive, negative.

9. `0` is unstable and `4` is stable.
10. Because setting the object temperature equal to the ambient temperature makes the derivative `0`.
11. `1.1`.
12. `1.22`.

`Worked hint for 5`

This is the standard equation $y' = ky$. Its solution family is exponential because the derivative is proportional to the function itself.

`Worked hint for 24`

Euler's Method uses $nextvalue = currentvalue + h * slope$. Compute the slope from the differential equation at the current point before updating the `y`-value.

Chapter 12

1. A displacement or direction quantity.
2. Velocity.
3. The vectors are perpendicular.
4. `13`.
5. $\langle 4, 3, -1 \rangle$.
6. `1`.

`Worked hint for 4`

Use the three-dimensional distance formula as a vector length formula:

$|\langle a, b, c \rangle| = \sqrt{a^2 + b^2 + c^2}$.

`Worked hint for 13`

Differentiate each coordinate separately. Velocity and acceleration are found component by component, but the final answer should still be read as a vector.

Chapter 13

1. Circles centered at the origin.
2. Rapid change in function value.
3. $f_x = 3x^2y, f_y = x^3 + 2y$.
4. $f_x = e^x \cos y, f_y = -e^x \sin y$.
5. Change with respect to `x` while `y` is held fixed.

6. Change with respect to `y` while `x` is held fixed.
7. $\langle 2x, 6y \rangle$.
8. $\langle 2, -12 \rangle$.
9. $L(x, y) = 5 + 2(x - 1) + 4(y - 2)$.
10. $2\sqrt{2}$.
11. $17/5$.
12. `(2, -1)`.
13. Local minimum.
14. `(0, 0)`.
15. Saddle point.
16. Saddle at `(-1, 0)` and local minimum at `(1, 0)`.
17. Maximum at `(6, 6)` with value `36`.
18. Minimum at `(3, 3)` with value `18`.
19. Maximum at $(5/\sqrt{2}, -5/\sqrt{2})$, minimum at $(-5/\sqrt{2}, 5/\sqrt{2})$.

`Worked hint for 8`

First compute the gradient function, then evaluate at the point. Students often substitute too early and lose the structural meaning of the derivative.

`Worked hint for 24`

Write the constraint as the level set $g(x, y) = x + y = 12$ and set $\nabla f = \lambda \nabla g$. The symmetry of the constraint and objective should make the final point believable before any algebra is finished.

Chapter 14

1. A total over a two-dimensional region.
2. When the region has radial or circular symmetry.
3. Because density times tiny volume gives tiny mass.
4. `6`.
5. 4pi.
6. `6`.
7. `4`.
8. $M_x = \iint_R y\varrho dA$, $M_y = \iint_R \times \varrho dA$, and $(x bar, y bar) = (M_y/m, M_x/m)$.
9. $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.
10. $x = \varrho\sin\phi\cos\theta$, $y = \varrho\sin\phi\sin\theta$, $z = \varrho\cos\phi$.
11. Because the total probability over the full region must be `1`.

`Worked hint for 6`

Area in polar coordinates requires the extra factor `r`. If that factor is missing, the setup is not geometrically correct.

`Worked hint for 15`

Integrate with respect to `y` first so the density behaves like a constant during the inner step. Then interpret the result: a heavier right side means the balance point cannot stay centered.

Chapter 15

1. A vector assigned to each point.
2. Work done along the path.
3. Net flow through the surface.
4. A potential is $f(x, y) = x^2 + y^2$.
5. One potential is $f(x, y) = x^3 + 2y^2$.
6. Because the line integral depends only on endpoint values of the potential.
7. $r_u = \langle 1, 0, 1 \rangle$, $r_v = \langle 0, 1, 1 \rangle$, $r_u \times r_v = \langle -1, -1, 1 \rangle$.

`Worked hint for 2`

A work line integral asks how much of the force acts in the direction of motion. The path matters because the relevant force component changes with the path direction.

`Worked hint for 19`

Differentiate the parameterization one variable at a time. The cross product should be perpendicular to both tangent vectors, so use that fact as a quick sign check.

Chapter 16

1. Start at `(1, 0)` and move clockwise.
2. $(3t^2 - 1)/(2t)$.
3. $(3\sqrt{3}/2, 3/2)$.
4. $A = (1/2)intr^2dtheta$.
5. Curvature measures turning of the tangent direction.
6. $-2/\sqrt{3}$.
7. At $t = 0$ and $t = \pi$.
8. At $t = \pi/2$ and $t = 3pi/2$.
9. Horizontal.

`Worked hint for 7`

Compute dx/dt and dy/dt first, then divide in the correct order. A very common mistake is reversing the quotient or substituting the parameter value too early.

`Worked hint for 31`

For a circle, symmetry strongly suggests constant curvature before any algebra begins. Use that geometric expectation as a check when simplifying the formula.

Chapter 17

1. Position and velocity together determine the initial state.
2. $r^2 - 3r + 2 = 0$.
3. Amplitude is maximum displacement; period is time for one full cycle.
4. Initial-value problems specify starting state, while boundary-value problems specify conditions at different points.
5. $y' = v, v' = -4y - 0.5v$.
6. $y = 3\cos 2t - \sin 2t$.
7. $y = e^{-t}(C_1 \cos 3t + C_2 \sin 3t)$.
8. $\pi/2$.
9. $2\sin 5t$.
10. `2`.
11. Overdamped or critically damped is more plausible than underdamped.

`Worked hint for 10`

Use the initial conditions one at a time. The position condition usually fixes the cosine coefficient immediately, and the velocity condition then determines the sine coefficient.

`Worked hint for 23`

Think in terms of which sine or cosine modes vanish at both endpoints. The endpoint conditions are selecting special frequencies, not arbitrary ones.

Chapter 18

1. The interval is unbounded.
2. The integrand becomes unbounded at an endpoint.
3. $\lim_{(} b \to \text{ infinity}) \int_a^b f(x)dx$.
4. `1`.
5. Diverges.
6. `2`.
7. Diverges.

8. Converges.
9. Converges for $p > 1$.
10. Converges for $p < 1$.
11. Converges by comparison with $1/x^2$.
12. Diverges by comparison with `1/x`.
13. $N \geq 100$.
14. Infinite support does not prevent the total integral from being `1`.

`Worked hint for 8`

The antiderivative $\ln x$ grows without bound. Do not stop after finding the antiderivative; the convergence question is answered only after taking the limit.

`Worked hint for 19`

A bounded denominator comparison is enough. If a positive function is always below a known convergent benchmark, then its total accumulation cannot exceed a finite amount.

Strategy notes by topic

Derivative-analysis problems

If the question asks for behavior, do not report only derivative formulas. Convert the derivative into:

- sign information,
- interval conclusions,
- and graph language.

Integral-application problems

If the question asks for area, volume, mass, work, or force, the most common error is not integration itself. It is choosing the wrong local slice or the wrong quantity for the integrand. Sketch first.

Series problems

For convergence questions, always ask what the object is:

- a sequence of terms,
- or a sequence of partial sums.

This single distinction resolves many early confusions.

Multivariable and vector problems

Check what kind of object the answer should be:

- scalar,
- vector,
- line,
- plane,
- or geometric interpretation.

The correct type of answer is often the first clue to the correct method.

Parametric, polar, and curvature problems

Do not rush to eliminate the parameter. For many problems, the parameter is the cleanest route to slope, speed, orientation, and turning information.

Second-order differential-equation problems

Classify the characteristic roots before doing anything else. The root type often tells the physical story faster than the final constants do.

Improper-integral problems

Always ask two questions first: where is the interval unbounded, and where is the integrand unbounded? Those locations determine how the limit setup must be written.

Appendix H. Common Misconceptions and Repair Strategies

Long textbooks spend many pages not because they love repetition, but because students predictably misunderstand recurring ideas. This appendix collects high-value misconception patterns across the calculus sequence and gives short repairs that instructors, tutors, and self-studying readers can apply quickly.

How to use this appendix

Each entry has three parts:

- `Misconception`: the incorrect belief or habit,
- `Why it happens`: the conceptual confusion underneath it,
- `Repair strategy`: a practical way to rebuild the idea.

The aim is diagnostic clarity. Once the misunderstanding is named correctly, the fix is usually much shorter than the confusion.

Limits and continuity

Misconception: the limit is just the function value

`Why it happens`

In many early examples, evaluating the formula at the point and taking the limit give the same result, so students merge the ideas.

`Repair strategy`

Use holes and removable discontinuities early. Ask for:

- the nearby behavior,
- the actual function value,
- and whether they agree.

The distinction becomes much harder to forget once a graph has a visible hole.

Misconception: if the left and right values look close in a table, the limit definitely exists

`Why it happens`

Tables provide evidence, but they do not prove the rule behind the behavior.

`Repair strategy`

Treat tables as hypothesis generators. Then ask what algebraic or structural reason supports the observed pattern. This is a good place to distinguish evidence from proof.

Derivatives

Misconception: derivative means "plug into a formula"

`Why it happens`

Once rule-based differentiation begins, the geometric meaning can fade behind symbol manipulation.

`Repair strategy`

Regularly ask for an interpretation:

- slope,
- instantaneous velocity,
- marginal change,
- or local linear prediction.

The derivative should keep a job, not just a notation.

Misconception: if $f'(c) = 0$, then `c` is automatically a max or min

`Why it happens`

Students see many examples where turning points do occur at derivative zero and overgeneralize.

`Repair strategy`

Use $f(x) = x^3$ and other flat-but-not-extreme examples repeatedly. Sign changes of `f'` should be emphasized more than the equation $f'(c) = 0$.

Misconception: the chain rule is an extra multiplication step

`Why it happens`

The formula is remembered mechanically rather than as a statement about nested dependence.

`Repair strategy`

Have the student name the inner function and the outer function before writing any derivative. If they cannot do that, they are not ready to differentiate the expression cleanly.

Graphing and optimization

Misconception: the second derivative alone explains the entire graph

`Why it happens`

Students hear that the second derivative describes bending and may overvalue it.

`Repair strategy`

Separate the jobs clearly:

- `f'` controls increase/decrease and turning behavior,
- `f''` refines the shape by describing slope change.

Neither one replaces the other.

Misconception: optimization means solve $f'(x) = 0$ and stop

`Why it happens`

The most visible algebraic step is the derivative equation, so students mistake it for the whole problem.

`Repair strategy`

Break optimization into named stages:

1. define the quantity,
2. use the constraint,
3. determine the domain,
4. differentiate,
5. compare candidates,

6. interpret the result.

This turns optimization into a modeling workflow rather than a single symbolic reflex.

Integrals

Misconception: a definite integral is always physical area

`Why it happens`

Area examples are the first and most visually memorable integral applications.

`Repair strategy`

Use rate and accumulation problems very early. Ask for units every time. If the units are liters, dollars, joules, or kilograms, the student can no longer pretend the answer is literally square units.

Misconception: negative area does not make sense, so a negative integral must be wrong

`Why it happens`

Students mix geometric area with signed accumulation.

`Repair strategy`

Use motion or cash-flow examples. Negative contribution is easier to understand as backward motion or debt than as "negative area."

Misconception: indefinite and definite integrals are basically the same thing with or without bounds

`Why it happens`

The notations look similar, and both use the integral sign.

`Repair strategy`

Ask what kind of object is returned:

- an indefinite integral returns a family of functions,
- a definite integral returns a number.

That object-level distinction is the cleanest repair.

Integration techniques

Misconception: there is one best integration trick for every problem

`Why it happens`

Students often want a universal algorithm that removes the uncertainty of method choice.

`Repair strategy`

Frame Chapter 8 as diagnosis rather than rule collection. Require students to explain why a chosen method fits the structure before they compute.

Misconception: substitution is guessing

`Why it happens`

When presented too quickly, substitution can look like arbitrary variable invention.

`Repair strategy`

Use a checklist:

- identify an inside expression,
- check for its derivative,
- ask whether the result becomes simpler.

This makes substitution a pattern-recognition move rather than a guess.

Sequences and series

Misconception: if the terms go to zero, the series converges

`Why it happens`

The necessary condition is mistaken for a sufficient one.

`Repair strategy`

Return repeatedly to the harmonic series. It is the cleanest permanent counterexample.

Misconception: a Taylor polynomial is just a fancy approximation formula to memorize

`Why it happens`

If the derivative-building idea is skipped, the polynomial seems arbitrary.

`Repair strategy`

Build Taylor polynomials from derivative matching at the center. Then the coefficients stop looking magical and start looking inevitable.

Differential equations

Misconception: the differential equation is only an algebra problem in disguise

`Why it happens`

Symbolic solving techniques can overshadow the meaning of the equation.

`Repair strategy`

Ask first: what does the derivative depend on? That question pushes the reader back toward rate laws, slope fields, and model behavior.

Misconception: logistic growth is just exponential growth with more symbols

`Why it happens`

Both equations contain the current quantity itself, and students may miss the feedback factor.

`Repair strategy`

Track the sign and size of `1 - P/M`. That factor is the entire modeling difference: it inserts self-limitation.

Vectors and multivariable calculus

Misconception: a point and a vector are the same kind of object

`Why it happens`

Both are written with coordinates, and the notation can blur their roles.

`Repair strategy`

Keep the language precise:

- a point marks location,
- a vector records displacement, direction, or velocity.

Use diagrams whenever possible.

Misconception: a few matching paths prove a multivariable limit exists

`Why it happens`

Path testing is introduced as a fast tool for disproof, but students invert its logic.

`Repair strategy`

State the asymmetry explicitly:

- one mismatching path pair disproves existence,
- many matching paths still do not prove existence.

This should be repeated whenever path tests are taught.

Misconception: the gradient is just a vector of partial derivatives

`Why it happens`

The formula is computationally simple, so the geometry may be ignored.

`Repair strategy`

Always pair the computation with contour language: the gradient points uphill and is perpendicular to level curves. The geometry gives the formula a job.

Multiple integration and vector calculus

Misconception: `dA` and `dV` are symbolic leftovers

`Why it happens`

Repeated symbolic manipulation can detach the differentials from geometry.

`Repair strategy`

Ask what the tiny piece represents:

- a strip,
- a rectangle,
- a sector,
- a tiny box,
- or an oriented patch.

The differential element should be seen as part of the model.

Misconception: flux and circulation are interchangeable because both use vector fields

`Why it happens`

Both quantities appear in late calculus and involve orientation.

`Repair strategy`

Use physical verbs:

- circulation measures "moving along",
- flux measures "passing through."

Those verbs reduce later sign confusion.

Teaching and self-study note

A misconception is rarely repaired by saying "that is wrong" more loudly. Repair usually requires one of three moves:

- separating two ideas that were fused,
- attaching a geometric or physical job to a formula,
- or supplying one decisive counterexample.

Calculus grows much faster once those repairs become deliberate rather than accidental.

Appendix I. Cumulative Review Problem Banks

This appendix is a large-bank companion for instructors, tutors, and self-studying readers who want more practice than the chapter exercise sets provide. The banks are cumulative within each part of the book, so later sections assume earlier material.

How to assign these banks

- Use `Skill refresh` problems for short homework or recitation warm-ups.
- Use `Mixed core` problems for weekly assignments.
- Use `Challenge and synthesis` problems for honors sections, exam review, or take-home sets.
- Use `Modeling and explanation` prompts when written interpretation matters as much as computation.

Part I Review Bank: Change and Derivatives

Skill refresh

1. Compute the average rate of change of $f(x) = x^2 - 1$ on `[1, 3]`.
2. Estimate $\lim_{x \to 2}(x^2 + 3x - 1)$.
3. Compute $\lim_{x \to -1}(x^2 - 4)/(x + 2)$.
4. Find the derivative of $f(x) = x^4 - 2x + 7$.
5. Find the derivative of $g(x) = (3x - 1)^5$.
6. Differentiate $h(x) = x^2 e^x$.
7. Differentiate $p(x) = \ln(x^2 + 1)$.
8. Differentiate $q(x) = \sin(4x^2)$.
9. Find the tangent line to $f(x) = x^2$ at $x = 3$.
10. Use linearization to estimate $\sqrt{9.1}$.

Mixed core

1. For $f(x) = x^3 - 6x$, determine intervals of increase and decrease.
2. Find the critical points of $g(x) = x^4 - 8x^2$.
3. Determine the concavity of $h(x) = x^3 - 3x^2$.
4. Determine whether $x = 0$ is an inflection point for $p(x) = x^3$.

5. Sketch a plausible graph of a function whose derivative is positive for $x < 0$, zero at $x = 0$, and negative for $x > 0$.
6. Find all local extrema of $r(x) = x/(x^2 + 1)$.
7. Differentiate the implicit relation $x^2 + xy + y^2 = 7$.
8. A particle has position $s(t) = t^3 - 6t^2 + 9t$. Find velocity and acceleration.
9. At what times in problem 18 is the particle at rest?
10. Use derivative evidence to determine when the particle in problem 18 moves left and right.

Challenge and synthesis

1. Explain why $f'(c) = 0$ is not enough to conclude that $f(c)$ is an extremum.
2. Construct a function with a critical point that is neither a maximum nor a minimum.
3. Construct a function with $f''(0) = 0$ but no inflection point at `0`.
4. Suppose `f'` is positive and increasing on an interval. What can be said about `f` and `f''` there?
5. Give an example of two different functions with the same derivative on an interval.
6. A company's revenue is $R(x) = x(120 - x)$. Find the production level that maximizes revenue.
7. A rectangle has perimeter `60`. Find the dimensions that maximize area.
8. A cylindrical can must hold a fixed volume. Explain what quantity would be optimized if the goal is to minimize material.

Modeling and explanation

1. Write a short explanation of why sign charts are interval tools rather than point tools.
2. Explain why tangent-line approximations become less reliable far from the base point.
3. A ladder slides down a wall. Describe how implicit differentiation enters the model before solving for a rate.
4. Describe a real context where concavity matters even if the function is always increasing.

Part II Review Bank: Integrals and Their Applications

Skill refresh

1. Interpret $\int_0^5 v(t)dt$ if `v` is velocity.
2. Compute $\int_0^3 (2x + 1)dx$.
3. Find an antiderivative of $6x^2 - 4x + 1$.
4. Solve $y' = 4x$ with $y(0) = 3$.

5. Differentiate $F(x) = \int_1^x (t^2 + 2)dt$.
6. Compute $int 2x \cos(x^2)dx$.
7. Compute $int \times e^x dx$.
8. Compute $int \sec^2 \times dx$.
9. Decompose and integrate $int 1/(x^2 - 1)dx$.
10. Estimate $\int_0^2 x^2 dx$ by the trapezoidal rule with two subintervals.

Mixed core

1. Explain why the Fundamental Theorem of Calculus connects derivatives and integrals.
2. Find the area between $y = x$ and $y = x^2$ on `[0, 1]`.
3. Find the volume generated by rotating $y = x$ on `[0, 1]` about the `x`-axis.
4. Find the volume generated by rotating the same region about the `y`-axis using shells.
5. Compute the mass of a rod on `[0, 3]` with density $\varrho(x) = 2 + x$.
6. Compute the center of mass of the rod in problem 15.
7. Set up the work integral for a force $F(x) = 5x$ moving an object from $x = 0$ to $x = 4$.
8. Explain why arc length uses a square root while area between curves does not.
9. Compute the surface area generated by rotating $y = x$ on `[0, 1]` about the `x`-axis.
10. Explain why numerical integration is often more realistic than symbolic integration in laboratory settings.

Challenge and synthesis

1. Give an example of a function whose definite integral on an interval is zero but which is not identically zero there.
2. Describe a situation where the shell method is clearly easier than washers.
3. Construct a physical interpretation of a negative definite integral.
4. Explain why substitution is the inverse structural partner of the chain rule.
5. Describe the denominator patterns that suggest partial fractions.
6. A tank fills at rate $r(t) = 3 + 0.5t$. Interpret and compute $\int_0^8 r(t)dt$.
7. A submerged plate has varying depth. List the steps required to construct a hydrostatic-force integral.
8. Explain the difference between total distance traveled and net displacement using integrals.

Modeling and explanation

1. Write a one-paragraph explanation of why accumulation is the unifying idea behind area, mass, and work.
2. Explain how units distinguish an indefinite integral from a definite integral in context.
3. Describe a data-based scenario where Simpson's Rule would be preferred over exact symbolic work.

4. A quantity decreases at a variable rate. Explain how an integral still reconstructs total change.

Part III Review Bank: Infinite Processes and Differential Equations

Skill refresh

1. Determine whether $a_n = 1/n$ converges.
2. Determine whether $b_n = (-1)^n$ converges.
3. State when a geometric series converges.
4. Find the sum of $3 + 3/2 + 3/4 + \ldots$.
5. Write the quadratic Taylor polynomial of e^x at $x = 0$.
6. Solve $dy/dx = 3x$.
7. Solve $dy/dx = 2y$.
8. Explain what a slope field represents.
9. Describe the carrying capacity in a logistic model.
10. Carry out one Euler step for $y' = x + y$, $y(0) = 1$, $h = 0.1$.

Mixed core

1. Explain why a series can diverge even if its terms approach zero.
2. Compute the first four partial sums of $1 + 1/2 + 1/4 + \ldots$.
3. Use a geometric-series formula to evaluate $1 + x + x^2 + \ldots$ for $|x| < 1$.
4. Use the linear Taylor polynomial of e^x to estimate $e^{0.1}$.
5. Use the quadratic Taylor polynomial of e^x to improve the estimate in problem 14.
6. Compare exponential growth and logistic growth in words.
7. Explain why Euler's Method is a local-linear algorithm.
8. Describe the long-term behavior of a logistic solution below carrying capacity.
9. Solve $dy/dx = xy$ by separation.
10. Use an initial condition to determine the constant in a separable-equation solution.

Challenge and synthesis

1. Give an example of a convergent sequence whose terms are not monotone.
2. Explain why partial sums, not terms, control series convergence.
3. Write a short explanation of how Taylor polynomials connect derivatives to approximation.
4. Describe a context where a logistic model is more realistic than an exponential model and explain why.

5. Explain why a slope field can be informative even when no elementary closed-form solution is available.
6. A model has equilibrium values `0` and `M`. Explain how derivative signs near those values predict long-term behavior.
7. Explain the difference between local numerical accuracy and long-term numerical reliability in Euler's Method.
8. A quantity doubles every fixed amount of time. Explain why this points toward an exponential model.

Modeling and explanation

1. Describe how you would use data to decide whether a process is better modeled by exponential or logistic growth.
2. Write a paragraph explaining why infinity in calculus is handled through limiting sequences rather than endless completed processes.
3. Explain how Taylor approximations support scientific computing.
4. Describe a real context where a quick Euler-style estimate is more useful than an exact formula.

Part IV Review Bank: Space, Multivariable Calculus, and Vector Calculus

Skill refresh

1. Find the length of $\langle 3, 4, 12 \rangle$.
2. Find the displacement from `(1, 0, 2)` to `(5, 3, 1)`.
3. Differentiate $r(t) = \langle t^2, \sin t, e^t \rangle$.
4. Compute $\langle 1, 2, 3 \rangle \cdot \langle 4, 0, -1 \rangle$.
5. State the geometric meaning of `u x v`.
6. Explain what a contour line represents.
7. Compute f_x and f_y for $f(x, y) = x^2 y + y^3$.
8. Find ∇f for $f(x, y) = x^2 + y^2$.
9. Compute $\int_0^2 \int_0^3 1 \, dy \, dx$.
10. Explain what flux measures.

Mixed core

1. Write a vector and parametric equation for the line through `(1, 2, 3)` in direction $\langle 2, -1, 4 \rangle$.
2. Write the equation of the plane through `(1, -1, 2)` with normal $\langle 2, 3, -1 \rangle$.
3. Explain the difference between velocity and speed for a space curve.

4. Describe a function of two variables whose limit at `(0,0)` does not exist because of path dependence.
5. Explain why the gradient is perpendicular to level curves.
6. Set up a double integral over the triangular region `0 \le y \le x \le 1`.
7. Find the area of a disk of radius `2` using polar coordinates.
8. Explain why the factor `r` appears in polar area.
9. Describe the difference between circulation and flux.
10. Explain how Green's Theorem echoes the Fundamental Theorem of Calculus.

Challenge and synthesis

1. Give an example of a parametric curve that is not the graph of any function $z = f(x, y)$.
2. Explain why a line in space is naturally described by a direction vector while a plane is naturally described by a normal vector.
3. Describe how the tangent plane is the multivariable analogue of local linearization.
4. Explain why testing several paths can disprove but not prove a multivariable limit.
5. Describe a region where reversing the order of integration substantially simplifies the bounds.
6. Explain why changing coordinates requires a correction factor.
7. A vector field appears rotational. Explain why circulation would be a natural quantity to study.
8. Explain why orientation matters for both line integrals and surface integrals.

Modeling and explanation

1. Describe how a terrain contour map communicates steepness without displaying a full surface.
2. Explain how a double integral could represent total pollutant concentration across a lake surface.
3. Describe a physical context where a cross product naturally appears.
4. Write a paragraph comparing the major integral theorems of calculus as one local-to-global family.

Appendix J. Practice Midterms and Final Review

This appendix is designed for actual course use. The exams are deliberately balanced between computation, interpretation, and setup. Instructors can shorten them, split them, or use them as review packets.

Practice Midterm I: Limits and Derivatives

1. Estimate $\lim_{x\to 2}(x^2 + 3x - 1)$.
2. Compute $\lim_{x\to 2}(x^2 - 4)/(x - 2)$.
3. Explain the difference between continuity and differentiability.
4. Find the derivative of $f(x) = x^4 - 3x + 2$.
5. Differentiate $g(x) = (2x - 1)^6$.
6. Differentiate $h(x) = x^2e^x$.
7. Differentiate $p(x) = \ln(x^2 + 4)$.
8. Find the tangent line to $f(x) = x^2$ at $x = 2$.
9. Use a sign chart to analyze increasing and decreasing behavior of $f(x) = x^3 - 3x$.
10. Explain why $f'(0) = 0$ does not guarantee an extremum.

`Short answer key`

1. `9`
2. `4`
3. $4x^3 - 3$
4. $12(2x - 1)^5$
5. $2xe^x + x^2e^x$
6. $2x/(x^2 + 4)$
7. $y = 4x - 4$

Practice Midterm II: Integrals and Applications

1. Find an antiderivative of $6x^2 - 4x + 1$.
2. Solve $y' = 4x$ with $y(0) = 2$.
3. Compute $\int_0^2 3x^2dx$.

4. Differentiate $F(x) = \int_1^x (t^2 + 4)dt$.
5. Compute int$2x \cos(x^2)dx$.
6. Compute int $\times e^x dx$.
7. Compute the area between $y = x$ and $y = x^2$ on `[0, 1]`.
8. Set up the shell-method volume for rotating the region under $y = x$ on `[0, 1]` about the `y`-axis.
9. Explain why signed area is not always the same as geometric area.
10. A rod has density $\varrho(x) = 2 + x$ on `[0, 3]`. Write the mass integral and evaluate it.

`Short answer key`

1. $2x^3 - 2x^2 + x + C$
2. $y = 2x^2 + 2$
3. `8`
4. $x^2 + 4$
5. $\sin(x^2) + C$
6. $e^x(x - 1) + C$
7. $1/6$
8. $\int_0^3 (2 + x)dx = 21/2$

Practice Midterm III: Series and Differential Equations

1. Determine whether $a_n = 1/n$ converges.
2. Determine whether $b_n = (-1)^n$ converges.
3. State when a geometric series converges.
4. Find the sum of $3 + 3/2 + 3/4 + \ldots$.
5. Write the quadratic Taylor polynomial of e^x at `0`.
6. Explain why the harmonic series is a useful counterexample.
7. What does a slope field represent?
8. Solve $dy/dx = 2y$.
9. Explain the difference between exponential and logistic growth.
10. Perform one Euler step for $y' = x + y, y(0) = 1, h = 0.1$.

`Short answer key`

1. Yes, to `0`
2. No
3. $|r| < 1$
4. `6`
5. $1 + x + x^2/2$

6. $y = Ce^{2x}$
7. `1.1`

Practice Midterm IV: Multivariable and Vector Calculus

1. Find the length of $\langle 3, 4, 12 \rangle$.
2. Compute $\langle 1, 2, 3 \rangle \cdot \langle 4, 0, -1 \rangle$.
3. State the geometric meaning of a cross product.
4. Explain what a contour line represents.
5. Compute f_x and f_y for $f(x, y) = x^2 y + 3y^2$.
6. Find ∇f for $f(x, y) = x^2 + y^2$.
7. Write an iterated integral over the region `0 \le y \le x \le 1`.
8. Explain why the factor `r` appears in polar coordinates.
9. Describe the difference between flux and circulation.
10. Explain how Green's Theorem fits the boundary-versus-interior theme.

`Short answer key`

1. `13`
2. `1`
3. $f_x = 2xy, f_y = x^2 + 6y$
4. $\langle 2x, 2y \rangle$
5. $\int_0^1 \int_0^x f(x, y)\, dy\, dx$

Comprehensive Final Review

Part A: conceptual short response

1. Distinguish a limit from a function value.
2. Explain why the derivative is a local quantity.
3. Explain why the definite integral is an accumulation quantity.
4. Explain why sequence convergence and series convergence are different questions.
5. Explain how the gradient generalizes slope.

Part B: core computation

1. Differentiate $f(x) = (3x^2 + 1)^5$.
2. Differentiate $g(x) = \ln(x^2 + 9)$.
3. Compute $\int_0^1 2xe^{x^2} dx$.
4. Compute $intx \sin \times dx$.

5. Find the local extrema of $h(x) = x^3 - 3x$.
6. Find the quadratic Taylor polynomial of e^x at `0`.
7. Solve $dy/dx = 3y$ with $y(0) = 2$.
8. Compute f_x and f_y for $f(x, y) = x^2 + xy$.
9. Compute $\int_0^2 \int_0^1 1 \, dy \, dx$.

Part C: setup and interpretation

1. Set up the area between $y = x$ and $y = x^2$ on `[0, 1]`.
2. Set up the shell-method volume for rotating the region under $y = x$ on `[0, 1]` about the `y`-axis.
3. Explain how Euler's Method uses local linearity.
4. Describe a multivariable limit that fails because of path dependence.
5. Explain why orientation matters for a line integral.
6. Compare the Fundamental Theorem of Calculus with Green's Theorem in one short paragraph.

`Short answer key`

1. $30x(3x^2 + 1)^4$
2. $2x/(x^2 + 9)$
3. `e - 1`
4. $-x \cos x + \sin x + C$
5. local max at `(-1, 2)`, local min at `(1, -2)`
6. $1 + x + x^2/2$
7. $y = 2e^{3x}$
8. $f_x = 2x + y, f_y = x$
9. `2`

Instructor note

These exams are intentionally broad. A traditional hardcover textbook often feels longer partly because it contains many pages of assessment material, review questions, and cumulative synthesis. This appendix is designed to serve that role directly.

Appendix K. Part I Extended Practice

This appendix provides a larger Chapter 1-5 problem bank for instructors who want deeper repetition and for readers who want more than the main chapter sets. Problems are grouped by chapter, but they are intended to be mixed across sections once the core ideas are established.

Chapter 1. Quantities and Functions

1. A reservoir drops from `1800` liters to `1320` liters in `4` hours. Find the average rate of change in liters per hour.
2. A population rises from `24,000` to `31,500` over `6` years. Find the average rate of change in people per year.
3. A business records profit values $P(2) = 18$ and $P(5) = 39$, where profit is in thousands of dollars. Find the average rate of change of profit on `[2,5]`.
4. A cyclist covers `48` kilometers in `2.4` hours. Compute the average speed.
5. Explain the difference between a quantity and a rate using a temperature example.
6. A graph is increasing but flattening. Describe what that suggests about change without using derivative language.
7. A table shows equal output increases for equal input increases. What does that suggest about the model?
8. Construct a real-life example where average rate of change is positive but the quantity eventually decreases later.
9. Two functions share the same table values on five sample inputs. Explain why they still might be different functions.
10. Write a short description of a function that is best represented by words rather than a simple formula.

Chapter 2. Limits and Continuity

1. Compute $\lim_{x\to 3}(x^2 - 2x + 1)$.
2. Compute $\lim_{x\to -2}(x^2 - 4)/(x + 2)$.
3. Compute $\lim_{x\to 5}(3x - 7)/(x + 1)$.
4. Compute $\lim_{x\to 0}(x^2 + x)/(x)$.
5. Evaluate $\lim_{x\to 4}(\sqrt{x} - 2)/(x - 4)$.

6. Determine whether the function defined by $f(x) = 2$ for $x < 1$ and $f(x) = 5$ for `x \ge 1` has a two-sided limit at `1`.
7. Give an example of a function whose limit exists at a point but whose function value is different there.
8. Explain why continuity can fail even when a graph looks almost unbroken.
9. Construct a function with a removable discontinuity at $x = 2$.
10. Construct a function with a jump discontinuity at $x = 0$.
11. Explain why one-sided limits are necessary for piecewise rules.
12. A function satisfies $\lim_{x \to 3^-} f(x) = 7$ and $\lim_{x \to 3^+} f(x) = 7$, but $f(3) = -4$. Is the function continuous at `3`?

Chapter 3. Derivatives from First Principles

1. Use the limit definition to find the derivative of $f(x) = x^2$.
2. Use the limit definition to find the derivative of $f(x) = 3x + 2$.
3. For $f(x) = 1/x$, write the difference quotient but do not simplify it fully.
4. Explain why the derivative is a limit of secant slopes rather than a single secant slope.
5. A particle has position $s(t) = t^2 - 4t$. Find its velocity using derivative language.
6. Describe the geometric meaning of a negative derivative.
7. Give an example of a function that is continuous at a point but not differentiable there.
8. Explain why a corner can destroy differentiability.
9. For a position graph, explain the difference between average velocity and instantaneous velocity.
10. A tangent line is horizontal at $x = 4$. What does that say about the derivative there?

Chapter 4. Derivative Rules

1. Differentiate $f(x) = 5x^6 - 2x + 9$.
2. Differentiate $g(x) = (x^2 + 1)(x^3 - 4)$.
3. Differentiate $h(x) = (x^2 + 3)/(x - 2)$.
4. Differentiate $p(x) = (4x - 1)^7$.
5. Differentiate $q(x) = e^{5x}$.
6. Differentiate $r(x) = \ln(x^2 + 16)$.
7. Differentiate $s(x) = \cos(3x^2)$.
8. Differentiate implicitly: $x^2 + y^2 = 16$.
9. Find the linearization of $\sqrt{x}$ at $x = 9$.
10. Use linearization to estimate $1/9.8$.
11. Explain why the chain rule is really about nested dependence.
12. Explain why implicit differentiation is useful even if `y` is not solved for explicitly.

Chapter 5. Behavior and Optimization

1. For $f(x) = x^3 - 3x$, determine intervals of increase and decrease.
2. Find the critical points of $g(x) = x^4 - 4x^2$.
3. Determine the concavity of $h(x) = x^3 - 6x^2$.
4. Determine whether $x = 0$ is a local extremum for $p(x) = x^3$.
5. Determine whether $x = 0$ is an inflection point for $p(x) = x^3$.
6. Use the first derivative test to classify critical points of $r(x) = x^4 - 8x^2$.
7. Use the second derivative test to classify critical points of $m(x) = x^2 - 6x + 10$.
8. Sketch a plausible graph of a function that is increasing, then decreasing, then increasing.
9. Sketch a plausible graph of a function that is concave down for $x < 2$ and concave up for $x > 2$.
10. A rectangle has perimeter `80`. Find the dimensions that maximize area.
11. A farmer fences three sides of a rectangular region against a river using `120` meters of fence. Find the dimensions that maximize area.
12. Explain why optimization problems often feel hardest before the derivative is taken.
13. A function has $f'(x) > 0$ and $f''(x) < 0$ on an interval. Describe the graph there in words.
14. A cost function is minimized at $x = 40$. Explain why a derivative zero alone does not fully justify that conclusion.

Mixed synthesis for Part I

1. Write a short explanation connecting limits, derivatives, tangent lines, and local linearity.
2. Give a real context where continuity matters more than differentiability.
3. Describe a situation where a derivative gives useful information even when a global formula for the function is unknown.
4. Create a piecewise function whose one-sided derivatives at a joining point differ.

Appendix L. Part II Extended Practice

This appendix extends the Chapter 6-9 exercise inventory. The problems are organized to support assignment building in integration-heavy parts of the course.

Chapter 6. Definite Integrals and Accumulation

1. Interpret $\int_0^4 r(t)dt$ if $r(t)$ is a water-flow rate in liters per minute.
2. Compute $\int_0^2 (x + 3)dx$.
3. Compute $\int_1^4 2x\,dx$.
4. Estimate $\int_0^3 x^2 dx$ using a right-endpoint Riemann sum with `3` equal subintervals.
5. Estimate the same integral using left endpoints.
6. Explain why Riemann sums become more accurate with finer partitions.
7. A velocity function is negative on an interval. Explain what that means physically.
8. Compute the average value of $f(x) = x^2$ on `[0, 2]`.
9. Explain why the average value of a function need not be one of the function values shown in a table.
10. A rate function changes sign. Explain why net change and total accumulated amount may differ.

Chapter 7. Antiderivatives and the Fundamental Theorem

1. Find an antiderivative of $4x^3 - 6x$.
2. Find an antiderivative of $e^x + \cos x$.
3. Solve $y' = 6x^2$ with $y(1) = 4$.
4. Compute $\int_0^3 2x\,dx$.
5. Differentiate $F(x) = \int_2^x (t^3 + 1)dt$.
6. Differentiate $G(x) = \int_0^{x^2} \cos t\,dt$.
7. Compute $int 3x^2 \sin(x^3)dx$.
8. Compute $\int_0^1 2x e^{x^2} dx$.
9. Explain why the constant of integration appears in indefinite integrals but not in definite integrals.

10. Explain the two main statements of the Fundamental Theorem of Calculus in words.

Chapter 8. Integration Techniques

1. Compute $int \times e^{x} dx$.
2. Compute $int x \cos \times dx$.
3. Compute $int \ln \times dx$.
4. Compute $int \sec^{2} \times dx$.
5. Compute $int \sin x \cos \times dx$.
6. Explain which trigonometric identity is useful for $int \sin^{2} \times dx$.
7. Describe the radical patterns that suggest trigonometric substitution.
8. Decompose and integrate $int 1/(x^{2} - 1) dx$.
9. Decompose and integrate $int 1/((x - 2)(x + 1)) dx$.
10. Explain when numerical integration should be preferred to exact symbolic work.
11. Estimate $\int_0^4 f(x) dx$ from a table of equally spaced values using the trapezoidal rule.
12. Compare the roles of substitution and integration by parts.

Chapter 9. Applications of Integration

1. Find the area between $y = x$ and $y = x^{2}$ on `[0, 1]`.
2. Set up the area between $x = y^2$ and $x = y + 2$.
3. Find the volume generated by rotating $y = x$ on `[0, 2]` about the `x`-axis.
4. Set up the shell-method volume for rotating the same region about the `y`-axis.
5. Compute the arc length of $y = 3x$ on `[0, 2]`.
6. Compute the surface area generated by rotating $y = x$ on `[0, 1]` about the `x`-axis.
7. Compute the mass of a rod on `[0, 3]` with density $\varrho(x) = 2 + x$.
8. Compute the center of mass of the rod in problem 39.
9. Set up a work integral for $F(x) = 5x$ from $x = 0$ to $x = 2$.
10. Describe the setup for a hydrostatic-force problem on a vertical plate.

Mixed synthesis for Part II

1. Explain how area, volume, work, and mass all come from the same slicing logic.
2. Give an example of a negative definite integral with a sensible physical interpretation.
3. Describe a situation where shells are easier than washers.
4. Explain why substitution is really a recognition method, not a trick.
5. Write a short paragraph comparing a definite integral with an antiderivative family.
6. Explain why units are one of the fastest checks for a correct integral setup.

Appendix M. Part III Extended Practice

This appendix extends the Chapter 10-11 material on infinite processes, approximation, and differential equations.

Chapter 10. Sequences, Series, and Taylor Approximation

1. Determine whether $a_n = 1/n$ converges.
2. Determine whether $b_n = (-1)^n$ converges.
3. Determine whether $c_n = n/(n+1)$ converges.
4. Explain why sequence convergence is about long-term term behavior.
5. Write the first five terms of the geometric series with first term `2` and ratio $1/3$.
6. Find the sum of $2 + 2/3 + 2/9 + \ldots$.
7. Explain why the harmonic series is a counterexample to the claim "terms going to zero imply convergence."
8. Compute the first four partial sums of $1 + 1/2 + 1/4 + \ldots$.
9. State the interval where $1 + x + x^2 + x^3 + \ldots$ converges.
10. Write the linear Taylor polynomial of e^x at `0`.
11. Write the quadratic Taylor polynomial of e^x at `0`.
12. Use the linear Taylor polynomial to estimate $e^{0.1}$.
13. Use the quadratic Taylor polynomial to improve the estimate in problem 12.
14. Explain why Taylor approximations are local rather than global.
15. Give an example of a function that is well approximated near one point but poorly approximated farther away.

Chapter 11. Differential Equations and Models

1. Explain what a slope field represents.
2. Solve $dy/dx = 4x$.
3. Solve $dy/dx = 3y$.
4. Solve $dy/dx = xy$ by separation.
5. Use $y(0) = 2$ to identify the constant in a separable solution.
6. A population satisfies $P' = 0.04P$ with $P(0) = 200$. Find $P(t)$.
7. Explain why exponential growth is not a good forever-model in most real systems.

8. Describe the carrying capacity in the logistic model.
9. Explain the sign of `P'` when $0 < P < M$ in a logistic model.
10. Explain the sign of `P'` when $P > M$ in a logistic model.
11. Perform one Euler step for $y' = x + y, y(0) = 1, h = 0.1$.
12. Perform a second Euler step for the same problem.
13. Explain why smaller step sizes usually improve Euler estimates.
14. Give a real situation where a logistic model is more realistic than an exponential model.
15. Explain why numerical methods are not side topics in differential equations.

Mixed synthesis for Part III

1. Explain why a convergent infinite process is still not a completed endless computation.
2. Describe how Taylor polynomials connect derivative information to practical estimation.
3. Explain why differential equations belong naturally in calculus rather than being a separate subject entirely.
4. Describe how slope fields, exact solutions, and numerical solutions answer different kinds of questions.
5. A model has equilibria at `0` and `M`. Explain how the derivative sign diagram predicts long-term behavior.
6. Give a short comparison between a power-series approximation and an Euler-style approximation.

Appendix N. Part IV Extended Practice

This appendix extends the Chapter 12-15 material on vectors, multivariable functions, multiple integration, and vector calculus.

Chapter 12. Vectors and Space

1. Find the length of $\langle 3,4,12 \rangle$.
2. Find the displacement from `(1, 2, 0)` to `(4, 3, 5)`.
3. Write a unit vector in the direction of $\langle 2,0,0 \rangle$.
4. Write a vector equation for the line through `(1, 2, 3)` in direction $\langle 2,-1,4 \rangle$.
5. Write the parametric equations of that line.
6. Write the equation of the plane through `(1, -1, 2)` with normal $\langle 2,3,-1 \rangle$.
7. Differentiate $r(t) = \langle t^2, \sin t, e^t \rangle$.
8. Find the speed of $r(t) = \langle \cos t, \sin t, t \rangle$.
9. Explain the difference between speed and velocity for a space curve.
10. Compute $\langle 1,2,3 \rangle \cdot \langle 4,0,-1 \rangle$.
11. State what it means if $u \cdot v = 0$.
12. Explain the geometric meaning of `u x v`.

Chapter 13. Multivariable Functions

1. Explain what a contour line represents.
2. Compute f_x and f_y for $f(x,y) = x^2y + 3y^2$.
3. Compute ∇f for $f(x,y) = x^2 + y^2$.
4. Evaluate the gradient of the function in problem 15 at `(1, 2)`.
5. Explain why multivariable limits are more subtle than one-variable limits.
6. Describe a path test that disproves the limit of $(xy)/(x^2 + y^2)$ at `(0,0)`.
7. Write the tangent-plane approximation for $f(x,y) = x^2 + y^2$ at `(1,1)`.
8. Explain why the gradient is perpendicular to level curves.
9. Describe a real context where a tangent plane functions as a local estimate.

Chapter 14. Multiple Integration

1. Compute $\int_0^2 \int_0^3 1 \, dy \, dx$.
2. Write an iterated integral over the triangular region `0 \le y \le x \le 1`.
3. Explain why reversing the order of integration can simplify a problem.
4. Find the area of a disk of radius `2` using polar coordinates.
5. Explain why the area element in polar coordinates is `r \, dr \, dtheta`.
6. Describe a region where polar coordinates are clearly more natural than rectangular coordinates.
7. Explain what a triple integral can represent physically.
8. Explain why coordinate changes need correction factors.
9. A density is larger farther from the origin in a disk. Explain why polar coordinates are a natural choice.

Chapter 15. Vector Calculus

1. Explain what a vector field is.
2. Explain the difference between a scalar field and a vector field.
3. Describe a field $F(x, y) = \langle x, y \rangle$ geometrically.
4. Describe a field $F(x, y) = \langle -y, x \rangle$ geometrically.
5. Explain what a line integral measures in a work setting.
6. Explain what flux measures across a surface.
7. Explain why orientation matters for line integrals.
8. Explain why orientation matters for surface flux.
9. State the main conceptual idea of Green's Theorem.
10. State the main conceptual idea of the Divergence Theorem.
11. State the main conceptual idea of Stokes' Theorem.
12. Explain how the major integral theorems echo the Fundamental Theorem of Calculus.

Mixed synthesis for Part IV

1. Explain how vectors supply the geometric language required by multivariable calculus.
2. Describe how gradient, divergence, and curl-like ideas each capture different kinds of local behavior.
3. Explain why line, surface, and multiple integrals all belong to one extended accumulation story.
4. Give one physical interpretation each for projection, flux, and circulation.

Appendix T. Section Mastery Banks

This appendix supplies additional chapter-by-chapter practice intended to move the manuscript toward the density of a mainstream classroom text. The problems are shorter than the end-of-chapter exercise sets, but they are organized to be assignable as daily section work, retrieval practice, quiz preparation, or cumulative review.

How to use this appendix

- Assign the first two or three items from a chapter as warm-up retrieval before class.
- Use the middle problems as section checks immediately after instruction.
- Use the later problems as bridge questions between routine computation and explanation.
- Pair these banks with the main chapter exercises when a full textbook workload is desired.

Chapter 1. Quantities, Functions, and the Shape of Change

1. A tank rises from `120` liters to `168` liters in `4` hours. Compute the average rate of change and include units.
2. Explain why the statement "the function changed by `5`" is incomplete without an interval.
3. A ride-share fare is modeled by $C(m) = 4 + 1.8m$. Interpret the meaning of the intercept and slope.
4. Give one example of a quantity whose graph would reasonably be piecewise linear over a short interval but not over a long interval.
5. A cyclist travels `15` miles in `50` minutes. Express the average speed in miles per hour.
6. Sketch two functions with the same average rate of change on `[0, 4]` but visibly different local behavior.
7. Explain why a table, a graph, and a formula can describe the same function while emphasizing different information.
8. Write one sentence describing a setting in which local linear thinking is useful before calculus is formalized.

Chapter 2. Nearness, Limits, and Continuity

1. Estimate $\lim_{(x \to 3)}(x^2 - 4)$ directly from continuity.
2. Evaluate $\lim_{(x \to -1)}(2x + 5)$.
3. Explain why a hole in a graph can still allow a two-sided limit to exist.
4. Give an example of a function that has a jump discontinuity at $x = 2$.
5. Explain the difference between the function value $f(2)$ and the limit $\lim_{(x \to 2)} f(x)$.
6. Compute $\lim_{(x \to 2)}(x^2 - 4)/(x - 2)$.
7. Explain why checking only integer inputs near a point is weak evidence for a limit.
8. Write a short paragraph distinguishing removable and nonremovable discontinuities.

Chapter 3. Derivatives

1. Describe the derivative as a limit of average rates of change.
2. Use the derivative definition to compute the derivative of $f(x) = x^2$ at $x = 1$.
3. Explain geometrically what a negative derivative means.
4. Give an example of a function that is continuous but not differentiable at a point.
5. Interpret the derivative of position as velocity in one sentence.
6. Explain why the derivative itself is a function and not only a number.
7. Describe one situation in which a second derivative is physically meaningful.
8. Sketch a graph whose derivative is positive at one point, zero at another, and undefined at a third.

Chapter 4. Working with Derivatives

1. Differentiate $x^5 - 3x + 1$.
2. Differentiate $(2x - 1)^6$.
3. Differentiate x^2e^x.
4. Differentiate $\ln(x^2 + 4)$.
5. Explain why the chain rule is needed for $\sin(3x^2)$.
6. Use implicit differentiation on $x^2 + y^2 = 25$.
7. Find the linearization of $\sqrt{x}$ at $x = 4$.
8. Describe one kind of error that linear approximation often makes when used too far from the base point.

Chapter 5. What Derivatives Tell Us

1. State the first-derivative test in plain language.
2. Find the critical points of $f(x) = x^3 - 3x$.

3. Determine where $f(x) = x^3 - 3x$ increases and decreases.
4. Explain why $f'(a) = 0$ does not guarantee a local extremum.
5. Find `f''(x)` for $f(x) = x^4 - 4x^2$.
6. Describe what concave up means geometrically.
7. Set up an optimization problem involving minimum fencing or packaging material.
8. Explain why every optimization answer should be translated back into context and units.

Chapter 6. The Integral as Accumulation

1. Explain the difference between signed area and total area.
2. Write a left-endpoint Riemann sum with `4` subintervals for $f(x)$ on `[0, 8]`.
3. State the geometric meaning of a definite integral.
4. Compute $\int_0^2 3dx$.
5. Compute $\int_0^3 \times dx$.
6. Explain why negative integrand values contribute negatively to net accumulation.
7. Describe how a Riemann sum becomes a definite integral conceptually.
8. Write one sentence explaining average value over an interval.

Chapter 7. Antiderivatives and the Fundamental Theorem

1. Find an antiderivative of `6x`.
2. Explain why antiderivatives differ by constants.
3. State the Fundamental Theorem of Calculus in words.
4. Compute $\int_1^3 2x dx$.
5. Solve the initial-value problem $F'(x) = 4x, F(0) = 7$.
6. Use substitution to evaluate $\int_0^1 2x(x^2 + 1)^3 dx$.
7. Explain why substitution is a recognition strategy rather than a memorized ritual.
8. Describe how an accumulation function $A(x) = \int_a^x f(t)dt$ changes when `f` is positive.

Chapter 8. More Integration Tools

1. Evaluate $int \times e^x dx$.
2. Evaluate $int \ln \times dx$.
3. Explain when integration by parts is usually more natural than substitution.
4. Evaluate $int \sin^3 \times \cos \times dx$.
5. Decompose $(3x + 5)/(x^2 + 3x + 2)$ into partial fractions.
6. Explain why numerical integration is useful even when exact methods exist in principle.

7. Compute one trapezoidal-rule estimate for $\int_0^2 f(x)dx$ using the values $f(0) = 1$, $f(1) = 2, f(2) = 5$.
8. Describe one signal that trigonometric substitution may be the right tool.

Chapter 9. Applications of Integration

1. Set up the area between $y = x$ and $y = x^2$ on `[0, 1]`.
2. Explain why a shell method problem starts with radius and height.
3. Set up a disk-method volume for rotating $y = \sqrt{x}$ on `[0, 4]` about the `x`-axis.
4. State the arc-length formula for $y = f(x)$.
5. Explain why surface area involves a square root factor beyond ordinary area.
6. Write an integral that would compute the mass of a rod with density $\varrho(x)$.
7. Describe how units help detect a flawed work or mass integrand.
8. Compare slicing and shelling in one short paragraph.

Chapter 10. Sequences and Series

1. Determine whether the sequence `1/n` converges.
2. Explain why $a_n \to 0$ is necessary for $\mathrm{sum} a_n$ to converge.
3. Classify $\mathrm{sum}(1/3)^n$.
4. Test $\mathrm{sum} 1/n$ for convergence.
5. Explain when the ratio test is a strong first choice.
6. Give an example of an alternating series.
7. Write the Maclaurin polynomial of degree `3` for e^x.
8. Explain the practical meaning of a Taylor remainder estimate.

Chapter 11. Differential Equations and Models

1. Describe what a slope field represents.
2. Solve $dy/dx = 2x$.
3. Solve $dy/dx = ky$.
4. Explain the role of carrying capacity in the logistic model.
5. Identify the equilibria of $y' = y(4 - y)$.
6. Describe the difference between model error and numerical error.
7. Compute one Euler step for $y' = x + y, y(0) = 1, h = 0.1$.
8. Explain why phase lines help even before a formula is found.

Chapter 12. Vectors and Space

1. Find the length of $\langle 3, 4, 12 \rangle$.
2. Write a vector equation for the line through `(1, 2, 3)` in direction $\langle 2, -1, 4 \rangle$.
3. Write the equation of the plane through `(1, 0, -2)` with normal vector $\langle 3, 1, 5 \rangle$.
4. Differentiate $r(t) = \langle t, t^2, t^3 \rangle$.
5. Explain the difference between velocity and speed.
6. Compute the dot product of $\langle 2, 1, 0 \rangle$ and $\langle 1, -3, 4 \rangle$.
7. Compute the cross product of $\langle 1, 0, 0 \rangle$ and $\langle 0, 1, 0 \rangle$.
8. Explain geometrically what projection means.

Chapter 13. Multivariable Functions

1. Describe a contour line in words.
2. Explain why matching along one path does not prove a multivariable limit exists.
3. Compute the partial derivatives of $f(x, y) = x^2y + 3y^2$.
4. Find the gradient of $f(x, y) = x^2 + y^2$.
5. Explain why the gradient is perpendicular to a level curve.
6. Write the tangent-plane approximation formula for $z = f(x, y)$ at `(a, b)`.
7. State the second-derivative test ingredients for a critical point.
8. Describe the idea behind Lagrange multipliers in one paragraph.

Chapter 14. Multiple Integration

1. Write a double integral over the rectangle $0 \leq x \leq 2, 0 \leq y \leq 3$.
2. Explain when changing the order of integration is helpful.
3. State the polar-area element for double integrals.
4. Write the Jacobian factor for cylindrical coordinates.
5. Explain why spherical coordinates suit radial symmetry.
6. Describe the difference between mass density and probability density in a double or triple integral.
7. Set up a double integral for the mass of a lamina with density $\varrho(x, y)$.
8. Explain what a triple integral accumulates conceptually.

Chapter 15. Vector Calculus

1. Define a vector field.
2. Explain the meaning of a line integral of work.
3. State Green's Theorem in words.

4. Distinguish circulation from flux.
5. Explain what makes a vector field conservative.
6. Describe the role of a normal vector in a surface integral.
7. Compare Stokes' Theorem and the Divergence Theorem in one short paragraph.
8. Explain why orientation matters in vector calculus.

Chapter 16. Parametric Curves, Polar Coordinates, and Curvature

1. Give a parametrization of the unit circle traced clockwise once.
2. Explain why a parametrized curve carries more information than the eliminated rectangular equation.
3. Find dy/dx for $x = t^2 + 1, y = t^3 - t$.
4. State the arc-length formula for a parametrized plane curve.
5. Convert $(r, \theta) = (3, \pi/6)$ to rectangular coordinates.
6. Explain why polar coordinates are not unique.
7. State the polar-area formula and describe where the factor $1/2$ comes from.
8. Explain what curvature measures that slope does not.

Chapter 17. Second-Order Differential Equations and Oscillation

1. Explain why a second-order equation typically needs two initial conditions.
2. Write the characteristic equation for $y'' - 3y' + 2y = 0$.
3. State the three root-type cases for constant-coefficient equations.
4. Explain the meaning of amplitude and period in simple harmonic motion.
5. Describe the difference between overdamped and underdamped motion.
6. Explain resonance in one paragraph.
7. Contrast initial-value and boundary-value problems.
8. Rewrite $y'' = -4y - 0.5y'$ as a first-order system.

Chapter 18. Improper Integrals and Long-Tail Behavior

1. Explain why $\int_1^i nfinity 1/x^2 dx$ is improper.
2. State the limit definition of an improper integral over `[a, infinity)`.
3. Determine whether $\int_1^i nfinity 1/x dx$ converges.
4. Determine whether $\int_0^1 1/\sqrt{x} dx$ converges.

5. State the `p`-integral threshold on `[1, infinity)`.
6. Explain why a function approaching `0` is not enough by itself to guarantee a finite improper integral.
7. Describe how the integral test connects improper integrals and series.
8. Explain one way improper integrals appear in probability.

Cumulative study patterns

- Use Chapters 1-5 together when preparing for derivative-based modeling.
- Use Chapters 6-9 together when practicing setup choices for accumulation problems.
- Use Chapters 10-11 and 17 together when building an infinite-process and differential-equation toolkit.
- Use Chapters 12-16 together when working with curves, coordinates, and geometry in higher dimensions.

Appendix X. Section Quizzes and Chapter Tests

This appendix supplies short assessment material that can be assigned as in-class quizzes, take-home checks, recitation sheets, or chapter tests. The aim is to make the manuscript more adoption-ready by adding the kind of routine assessment infrastructure found in longer commercial texts.

Quiz design notes

- Each chapter quiz mixes one or two conceptual prompts with direct skill checks.
- The chapter tests are longer and mix setup, computation, and interpretation.
- Instructors can assign these as written or split them into smaller daily checks.

Chapter 1 quiz

1. Compute the average rate of change of a quantity rising from `18` to `45` over `9` minutes.
2. Explain why a rate without units is incomplete.
3. Interpret the slope in a linear cost model.
4. Sketch two graphs with the same average rate of change on the same interval.

Chapter 2 quiz

1. Compute $\lim_{(} x \to 2)(x^2 + x - 6)/(x - 2)$.
2. Distinguish left-hand and right-hand limits in words.
3. Give an example of a removable discontinuity.
4. Explain why continuity requires agreement between value and nearby behavior.

Chapter 3 quiz

1. State the derivative definition.
2. Interpret a negative derivative physically.
3. Explain why continuity does not guarantee differentiability.
4. Compute the derivative of x^2 at $x = 3$ from the known derivative formula.

Chapter 4 quiz

1. Differentiate $(x^2 + 1)^4$.
2. Differentiate $x^2 e^x$.
3. Use implicit differentiation on $x^2 + y^2 = 9$.
4. Explain when linearization is trustworthy.

Chapter 5 quiz

1. Find the critical points of $x^3 - 3x$.
2. Explain what concave up means.
3. State the first-derivative test in words.
4. Describe one context in which optimization appears naturally.

Chapter 6 quiz

1. Write a Riemann sum for $f(x)$ on `[0, 4]` using `4` equal subintervals.
2. Compute $\int_0^2 5dx$.
3. Explain signed area versus total area.
4. State the meaning of average value over an interval.

Chapter 7 quiz

1. Find an antiderivative of $4x^3$.
2. State the Fundamental Theorem of Calculus in words.
3. Evaluate $\int_0^1 2x\,dx$.
4. Explain why substitution works.

Chapter 8 quiz

1. Evaluate $int \times e^x dx$.
2. Explain how to choose between parts and substitution.
3. State one use of numerical integration.
4. Decompose a simple rational function into partial fractions.

Chapter 9 quiz

1. Set up area between $y = x$ and $y = x^2$ on `[0, 1]`.

2. Explain shell radius versus shell height.
3. State the arc-length formula for a graph.
4. Describe how a density function changes an ordinary integral into a mass integral.

Chapter 10 quiz

1. Explain why $a_n \to 0$ is necessary for series convergence.
2. Classify a geometric series with ratio $1/4$.
3. State when the ratio test is useful.
4. Write the degree-`2` Maclaurin polynomial for e^x.

Chapter 11 quiz

1. Explain what a slope field shows.
2. Solve $dy/dx = 3x$.
3. Identify equilibria of $y' = y(5 - y)$.
4. Compute one Euler step for $y' = x + y, y(0) = 1, h = 0.1$.

Chapter 12 quiz

1. Compute the length of $\langle 1, 2, 2 \rangle$.
2. Write a vector equation of a line through a point with a direction vector.
3. Explain speed versus velocity.
4. State one geometric use of the cross product.

Chapter 13 quiz

1. Compute f_x and f_y for $f(x, y) = x^2 y$.
2. Explain why the gradient is perpendicular to level curves.
3. Write the tangent-plane approximation formula.
4. Describe the idea of a saddle point.

Chapter 14 quiz

1. Write a double integral over a rectangle.
2. Explain why the factor `r` appears in polar coordinates.
3. State one reason to change order of integration.
4. Describe what a triple integral accumulates.

Chapter 15 quiz

1. Define a line integral of work.
2. Explain flux in words.
3. State one test for a conservative field.
4. Describe how the major vector-calculus theorems resemble the Fundamental Theorem of Calculus.

Chapter 16 quiz

1. Find dy/dx for a simple parametrized curve.
2. Explain why polar coordinates are not unique.
3. State the polar-area formula.
4. Describe what curvature measures.

Chapter 17 quiz

1. Write the characteristic equation for a constant-coefficient ODE.
2. Explain the difference between underdamped and overdamped motion.
3. State the general form of simple harmonic motion.
4. Contrast initial-value and boundary-value problems.

Chapter 18 quiz

1. Explain why $\int_1^i nfinity1/x^2dx$ is improper.
2. State the `p`-integral threshold on `[1, infinity)`.
3. Explain why `1/x` and $1/x^2$ have different tail behavior.
4. Describe one probability interpretation of an improper integral.

Test A. Limits and derivatives

1. Estimate a limit numerically and then compute it algebraically.
2. Explain why a function can be continuous but nondifferentiable.
3. Differentiate a composite expression.
4. Analyze intervals of increase and decrease from a derivative.
5. Solve an optimization problem and interpret the final answer in context.

Test B. Integrals and applications

1. Build a Riemann sum and explain what it approximates.
2. Use the Fundamental Theorem to evaluate a definite integral.
3. Choose and justify an integration technique.
4. Set up an area or volume application problem.
5. Explain how units validate or reject an application integrand.

Test C. Series and differential equations

1. Distinguish sequence convergence from series convergence.
2. Classify a geometric series and a nongeometric comparison-test example.
3. Use the ratio or root test on a factorial or power expression.
4. Solve a separable or exponential-growth differential equation.
5. Explain the behavior of an equilibrium in a logistic or autonomous model.

Test D. Space and multivariable calculus

1. Write equations for a line and a plane in space.
2. Interpret velocity, speed, and tangent direction for a space curve.
3. Compute a gradient and use it in a directional-derivative setting.
4. Set up a double or triple integral with a sensible coordinate system.
5. Explain why the tangent plane is a multivariable linear approximation.

Test E. Vector calculus and advanced topics

1. Explain circulation versus flux.
2. Identify a conservative field or explain why a field is not conservative.
3. Use parametric or polar reasoning to compute a slope or area.
4. Solve a second-order constant-coefficient differential equation.
5. Classify an improper integral using direct computation or comparison.

Oral-exam variants

- Ask students to explain a theorem in words before allowing formulas.
- Ask students to identify what kind of object the answer should be before solving.
- Ask students to give a reasonableness check after every numerical or symbolic answer.

Appendix Y. Extended Chapter Problem Banks

This appendix adds another layer of assignable problems beyond the main chapter exercises, homework sets, quizzes, and mastery banks. The goal is straightforward: mainstream calculus texts feel thick partly because every chapter offers far more problems than one semester will use. This appendix moves the manuscript in that direction.

Chapter 1

1. A car's odometer rises from `18,240` to `18,390` miles in `3` hours. Compute the average rate of change.
2. Build a table, formula, and graph description for a linearly cooling liquid.
3. Explain how a graph can show a constant rate even when no formula is given.
4. Give an example of a quantity whose units are dollars per hour and explain the meaning.
5. Sketch a function whose average rate of change on `[0, 4]` is zero but which is not constant.
6. Interpret the slope in a taxi-fare model with base fee plus per-mile charge.
7. Compare two functions that share the same value at $x = 2$ but different local behavior there.
8. Write a brief explanation of why local linearity is a modeling habit rather than only a graphing habit.

Chapter 2

1. Compute $\lim_{(} x \to 1)(x^2 + 3x - 4)/(x - 1)$.
2. Evaluate a one-sided limit for a step function.
3. Describe a function that is continuous from the right but not from the left at a point.
4. Explain why a function value can be changed without changing the limit.
5. Compute $\lim_{(} x \to 0)(\sin x)/x$ using known benchmark behavior.
6. Give a graphical description of a jump discontinuity.
7. Build a piecewise function with a removable discontinuity at $x = 3$.
8. Explain why continuity on an interval is stronger than continuity at one point.

Chapter 3

1. Use the definition to compute the derivative of $f(x) = 3x + 1$.
2. Explain the difference between instantaneous and average velocity.
3. Give an example of a point where the tangent is vertical.
4. Describe a graph that is continuous but has a corner.
5. Explain why higher derivatives matter in motion.
6. Estimate a tangent slope from nearby data values.
7. Compare secant-line slope and tangent-line slope in one paragraph.
8. Explain why a derivative can fail to exist at a cusp.

Chapter 4

1. Differentiate $x^7 - 2x^3 + 9$.
2. Differentiate $(5x - 4)^8$.
3. Differentiate $e^(3x^2)$.
4. Differentiate $\ln(\sin x)$.
5. Use implicit differentiation on $x^3 + y^3 = 6xy$.
6. Find the linearization of `1/x` at $x = 2$.
7. Explain when logarithmic differentiation is a good structure-reading tool.
8. Compare product rule and chain rule in a short example.

Chapter 5

1. Build a sign chart for $f'(x) = (x - 2)(x + 1)$.
2. Find intervals of increase and decrease for a polynomial with two critical points.
3. Explain why $f''(c) = 0$ does not force an inflection point.
4. Describe a function that is increasing and concave down on the same interval.
5. Solve an optimization problem involving minimum surface area of a box with fixed volume.
6. Explain why endpoint checking matters on closed intervals.
7. Compare the first-derivative and second-derivative tests.
8. Describe the difference between a local maximum and an absolute maximum.

Chapter 6

1. Write a midpoint Riemann sum with `5` subintervals for $f(x)$ on `[0, 10]`.
2. Compute $\int_1^4 (2x - 1)dx$.
3. Explain the effect of negative integrand values on net accumulation.

4. Compare signed area and geometric area for a graph crossing the axis.
5. Describe why partition refinement matters in Riemann sums.
6. State one property of definite integrals over adjacent intervals.
7. Explain average value of a function using the language of a balancing rectangle.
8. Build a rate-to-total story that leads naturally to a definite integral.

Chapter 7

1. Find $int(6x^5 - 2x + 1)dx.$
2. Solve $F'(x) = 3x^2 - 4, F(1) = 2.$
3. Evaluate $\int_0^2 x^3 dx.$
4. Use substitution on $intx(x^2 + 5)^4 dx.$
5. Explain why antiderivatives are families.
6. Describe an accumulation function whose derivative is negative on an interval.
7. Compare indefinite and definite integrals.
8. Explain the role of the constant of integration.

Chapter 8

1. Evaluate $int \times \cos \times dx.$
2. Evaluate $int \sec^2 \times dx.$
3. Evaluate $intx/(x^2 + 9)dx.$
4. Explain why $int \ln \times dx$ is an integration-by-parts problem.
5. Decompose `(2x + 7)/((x + 1)(x + 3))`.
6. Compare trapezoidal and Simpson reasoning in words.
7. Describe one warning sign that an exact antiderivative may be less useful than a numerical estimate.
8. Build a method-choice flow for substitution, parts, partial fractions, and trig identities.

Chapter 9

1. Set up the area between $y = x^2$ and $y = 2x$ on their intersection interval.
2. Write a shell-method setup for revolving a region about the `y`-axis.
3. State the washer-method formula in words.
4. Set up the mass of a variable-density rod on `[0, 5]`.
5. Set up the work needed to stretch a spring from $x = 1$ to $x = 3$.
6. Explain why fluid force depends on depth.
7. Compare surface area and arc length structurally.
8. Explain how units distinguish mass, work, and area integrals.

Chapter 10

1. Decide whether the sequence $(-1)^n$ converges.
2. Classify $sum2^(- n)$.
3. Use comparison on $sum1/(n^2 + 1)$.
4. Use the ratio test on $sumn!/5^n$.
5. Explain when the alternating-series test applies.
6. Write the Maclaurin polynomial of degree `4` for $\cos x$.
7. Describe the difference between absolute and conditional convergence.
8. Explain why Taylor polynomials are local approximations rather than global identities.

Chapter 11

1. Sketch the behavior of solutions to $y' = y(2 - y)$ using equilibrium reasoning.
2. Solve $dy/dx = x/y$.
3. Solve $P' = 0.2P, P(0) = 50$.
4. Explain the carrying capacity in the logistic model.
5. Compute two Euler steps for a simple initial-value problem.
6. Describe how a slope field reveals stable and unstable behavior.
7. Compare exponential growth with logistic growth.
8. Explain why numerical methods are unavoidable for many differential equations.

Chapter 12

1. Find a unit vector in the direction of $\langle 3, -4, 12 \rangle$.
2. Write parametric equations of a line through two points in space.
3. Find the equation of a plane from a point and normal.
4. Differentiate $r(t) = \langle \cos t, \sin t, t \rangle$.
5. Compute speed from a velocity vector.
6. Use the dot product to find the angle between two vectors.
7. Use the cross product to find the area of a parallelogram.
8. Explain why a curve in space may fail to be a graph of $z = f(x, y)$.

Chapter 13

1. Compute a multivariable limit that depends on path and explain why it fails to exist.
2. Find partial derivatives of a product expression.
3. Compute a directional derivative from a gradient.
4. Write the tangent plane to $z = x^2 + y^2$ at `(1, 2)`.

5. Find critical points of a quadratic form.
6. Classify a critical point as maximum, minimum, or saddle.
7. Set up a Lagrange multiplier problem with a circular constraint.
8. Explain why the gradient points normal to a level curve.

Chapter 14

1. Reverse the order of integration for a simple triangular region.
2. Set up a polar-coordinate integral over a disk sector.
3. Explain why polar coordinates simplify circular regions.
4. Write a triple integral over a rectangular box.
5. Compute the mass of a lamina with constant density over a rectangle.
6. Describe the meaning of a Jacobian factor.
7. Compare cylindrical and spherical coordinates.
8. Explain how probability density changes the interpretation of a multiple integral.

Chapter 15

1. Sketch or describe a radial vector field.
2. Write a line integral setup for work along a curve.
3. Explain why conservative fields give path independence.
4. Describe circulation in words.
5. Describe flux in words.
6. Explain when Green's Theorem can replace direct integration.
7. Compare Stokes' Theorem and the Divergence Theorem.
8. Explain why orientation errors change signs in vector-calculus answers.

Chapter 16

1. Parametrize a circle traced twice.
2. Find the tangent line to a simple plane parametric curve at a given parameter.
3. Determine where a parametrized curve has vertical tangents.
4. Convert a polar equation to rectangular form.
5. Sketch $r = 1 - \cos\theta$.
6. Set up the area enclosed by a rose curve petal.
7. Explain why arc length in polar coordinates uses both `r` and `dr/dtheta`.
8. Compare slope and curvature as local descriptors.

Chapter 17

1. Solve a constant-coefficient equation with distinct real roots.
2. Solve one with a repeated root.
3. Solve one with complex roots.
4. Compute amplitude or period for a harmonic motion model.
5. Explain the difference between overdamped and critically damped behavior.
6. Describe resonance in a real system.
7. Compare transient and forced response.
8. Explain why a second-order model naturally tracks both position and velocity.

Chapter 18

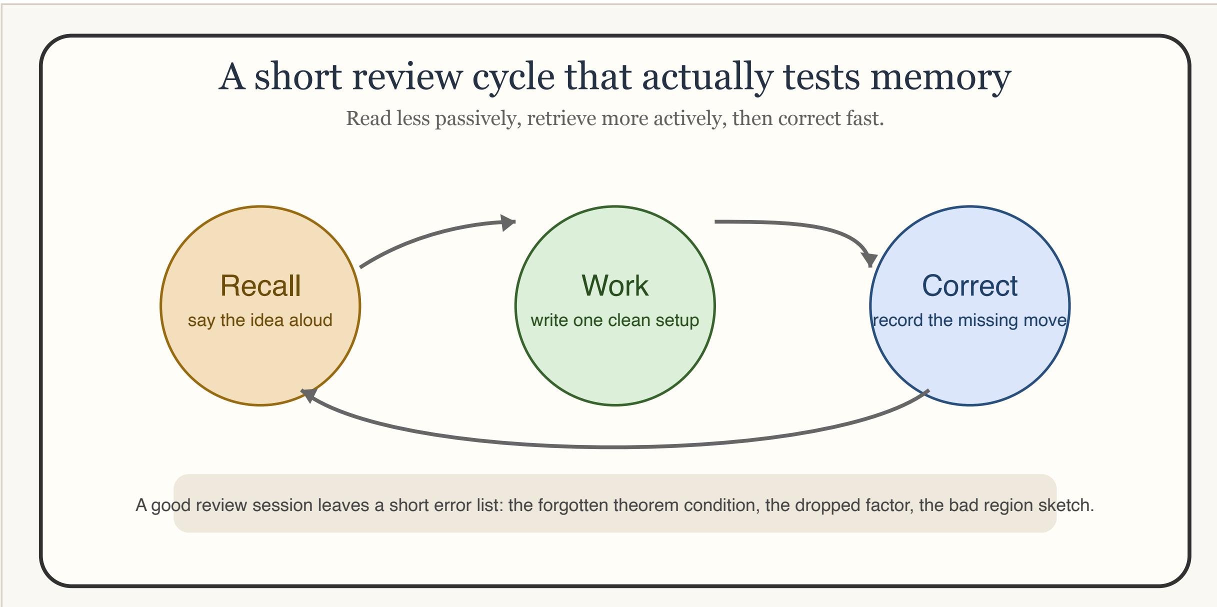
1. Rewrite an improper integral over `[1, infinity)` as a limit.
2. Classify $\int_1^{i} nfinity1/x^3dx$.
3. Classify $\int_0^1 1/x^{(}2/3)dx$.
4. Compare the tails of `1/x`, $1/x^2$, and $1/x^4$.
5. Use comparison to classify $\int_1^{i} nfinity1/(x^2 + 4x + 1)dx$.
6. Explain why an odd integrand on `(-infinity, infinity)` must still be handled carefully.
7. Use the integral test to discuss $sum 1/n^3$.
8. Explain one role of improper integrals in probability or statistics.

Assignment ideas

- Use this appendix for alternate homework when the main chapter set has already been assigned.
- Use Chapters 10, 11, 17, and 18 together for a stronger infinite-process and modeling track.
- Use Chapters 12 through 16 together for an extended geometry-and-coordinates recitation sequence.

Appendix AA. Board-Style Review and Oral Prompts

This appendix is designed for the last stage of preparation before a quiz, midterm, oral defense, tutoring session, or board presentation. The questions are short on purpose. Their job is to reveal whether the underlying structure is available without notes.



How to use this appendix

The strongest use pattern is:

1. read one prompt,
2. answer aloud without writing for thirty to sixty seconds,
3. write the key equation or picture on paper or a board,
4. compare with the answer cue,
5. record the missing move, not just the missing formula.

This appendix works well with the chapter review sheets and the worked-solution appendices:

- the review sheets compress the chapter,
- the oral prompts test whether that compression is really in memory,
- the worked examples rebuild a full solution when recall breaks down.

Part I. Seeing Change

Prompt 1. Average versus instantaneous change

Question: What is the difference between average rate of change on `[a, b]` and the derivative at `a`?

Strong answer cue: average rate compares total output change with total input change across an interval; the derivative is the limit of those nearby average rates and therefore measures local behavior at a point.

Prompt 2. A function value versus a limit

Question: How can $\lim_{(x \to a)} f(x)$ exist even when $f(a)$ is undefined?

Strong answer cue: a limit depends on nearby values, not necessarily the value at the point itself. A removable discontinuity is the standard example.

Prompt 3. Why continuity is not automatic

Question: Why can a piecewise function fail to be continuous even when each piece is a familiar formula?

Strong answer cue: continuity depends on how the pieces meet at the boundary. The left-hand limit, right-hand limit, and actual function value must agree.

Prompt 4. Derivative as a function

Question: Why is `f'(x)` usually treated as a new function instead of only a number?

Strong answer cue: every input point can have its own instantaneous rate, so differentiation builds a new rule that records local behavior across the whole domain.

Prompt 5. Local linearity

Question: What does it mean to say a function is locally linear near a point?

Strong answer cue: on a sufficiently small scale the function is well-approximated by its tangent line, so nearby changes can be estimated by a linear model.

Part II. Reading Behavior

Prompt 6. Critical points are candidates

Question: Why is $f'(c) = 0$ not enough by itself to prove a local maximum or minimum?

Strong answer cue: the derivative being zero only marks a candidate location. Classification still requires sign change, endpoint comparison, or a conclusive second-derivative test.

Prompt 7. Concavity and the second derivative

Question: What does $f''(x) > 0$ say geometrically?

Strong answer cue: the slope is increasing and the graph bends upward, so tangent lines tend to lie below the graph near that point.

Prompt 8. Definite integral meaning

Question: What is the shortest correct sentence that explains a definite integral?

Strong answer cue: it is the accumulated total of many local contributions, often interpreted as net change or signed area depending on context.

Prompt 9. Net change versus total amount

Question: Why can $\int_a^b v(t)dt$ and $\int_a^b |v(t)|dt$ answer different physical questions?

Strong answer cue: the first gives displacement or net signed change; the second gives total distance traveled or total accumulated magnitude.

Prompt 10. The Fundamental Theorem in words

Question: What is the conceptual content of the Fundamental Theorem of Calculus?

Strong answer cue: differentiation and accumulation reverse one another when the accumulated quantity is tracked correctly, so local rate information rebuilds global change and vice versa.

Part III. Building and Approximating

Prompt 11. Method choice in integration

Question: What should you inspect before deciding between substitution, parts, partial fractions, and a numerical method?

Strong answer cue: the structure of the integrand. Look for an inside derivative, product simplification, rational-factor decomposition, or the absence of a practical elementary antiderivative.

Prompt 12. Slices and shells

Question: Why are washer and shell methods really one idea?

Strong answer cue: both accumulate the contributions of thin pieces. The difference is geometric orientation and which local formula makes the slice easier to describe.

Prompt 13. Sequence versus series

Question: What is the fastest correct way to distinguish a sequence from a series?

Strong answer cue: a sequence is a list of terms; a series is the limit behavior of partial sums formed from those terms.

Prompt 14. Why term limits are not enough

Question: Why does $a_n \to 0$ fail to guarantee that $\sum a_n$ converges?

Strong answer cue: the terms can shrink too slowly, so the partial sums can still grow without bound. The harmonic series is the standard model.

Prompt 15. Taylor models

Question: What information do Taylor polynomials use to approximate a function?

Strong answer cue: derivative data at a chosen center point. The polynomial matches the function's local value, slope, and higher-order behavior there.

Part IV. Calculus in More Than One Direction

Prompt 16. Vectors versus points

Question: What is the practical distinction between a point and a vector in space?

Strong answer cue: a point locates position, while a vector records displacement, direction, or velocity independently of location.

Prompt 17. The gradient

Question: What does the gradient tell you that the two partial derivatives alone do not?

Strong answer cue: it packages steepest increase direction and maximum rate of increase into one vector object.

Prompt 18. Directional derivatives

Question: Why must the direction vector be normalized when computing a directional derivative?

Strong answer cue: otherwise the answer would depend on both direction and length. A directional derivative should measure rate per unit step in the given direction.

Prompt 19. Coordinate choice

Question: When should polar or cylindrical coordinates be your first instinct?

Strong answer cue: when the region, integrand, or boundary has circular or radial symmetry that becomes simpler in those coordinates.

Prompt 20. The Jacobian idea

Question: What is the geometric role of the extra factor in a change-of-variables formula?

Strong answer cue: it corrects for how the coordinate transformation stretches or compresses area or volume elements.

Part V. Oscillation, Coordinates, and Turning

Prompt 21. Parametric curves

Question: What can a parametrization tell you that a simple graph $y = f(x)$ may hide?

Strong answer cue: order of tracing, repeated motion, direction, speed, and curves that fail the vertical-line test.

Prompt 22. Curvature

Question: What does curvature measure conceptually?

Strong answer cue: how rapidly the direction of the tangent changes with respect to arc length, not just how steep the graph looks.

Prompt 23. Oscillation from roots

Question: Why do complex characteristic roots lead to oscillation?

Strong answer cue: complex roots generate sine and cosine terms, which encode repeated sign-changing motion around equilibrium.

Prompt 24. Damping

Question: What does damping do to a second-order system?

Strong answer cue: it removes energy from the system, reducing amplitude over time and sometimes preventing continued oscillation.

Part VI. Infinite Domains and Long Tails

Prompt 25. Improper integral as a limit

Question: Why is it dangerous to integrate an improper integral without first writing the limit?

Strong answer cue: the convergence issue is the main question. The formal antiderivative step is not enough unless the limiting process is valid.

Prompt 26. Comparison logic

Question: What is the one sentence you should say before using comparison on an improper integral?

Strong answer cue: identify a benchmark function with known behavior that dominates or is dominated by the given integrand for large `x` or near the singularity.

Prompt 27. Finite area over an infinite interval

Question: How can a graph stretch forever and still enclose finite area?

Strong answer cue: if the tail decays fast enough, the remaining contributions become arbitrarily small and the limiting total settles to a finite number.

Error-diagnosis drills

Each of the following is a common board-work error pattern. State the repair out loud before computing.

Drill 1. Missing inner derivative

Broken line: $d/dx[(x^2 + 1)^5] = 5(x^2 + 1)^4$.

Repair cue: the chain rule is incomplete; multiply by the derivative of the inside, `2x`.

Drill 2. Wrong top-minus-bottom order

Broken line: $\text{Area} = \int_0^1 (x^2 - x)dx$.

Repair cue: on `[0, 1]`, `x` lies above x^2, so the integrand must be $x - x^2$.

Drill 3. Series terms versus series sums

Broken line: "$a_n \to 0$, so the series converges."

Repair cue: term limits are necessary, not sufficient. Test the partial sums with comparison, ratio, root, integral, or another appropriate tool.

Drill 4. Missing `r` in polar coordinates

Broken line: $\iint_R f(x, y)dA = \iint f(r\cos\theta, r\sin\theta) dr\, dtheta$.

Repair cue: area stretches under polar coordinates, so $dA = r\, dr\, dtheta$.

Drill 5. Flux-circulation mix-up

Broken line: "The line integral around the curve measures flux."

Repair cue: a line integral of `F . dr` measures circulation or work along the path; flux needs a normal direction across a boundary.

A seven-day review schedule

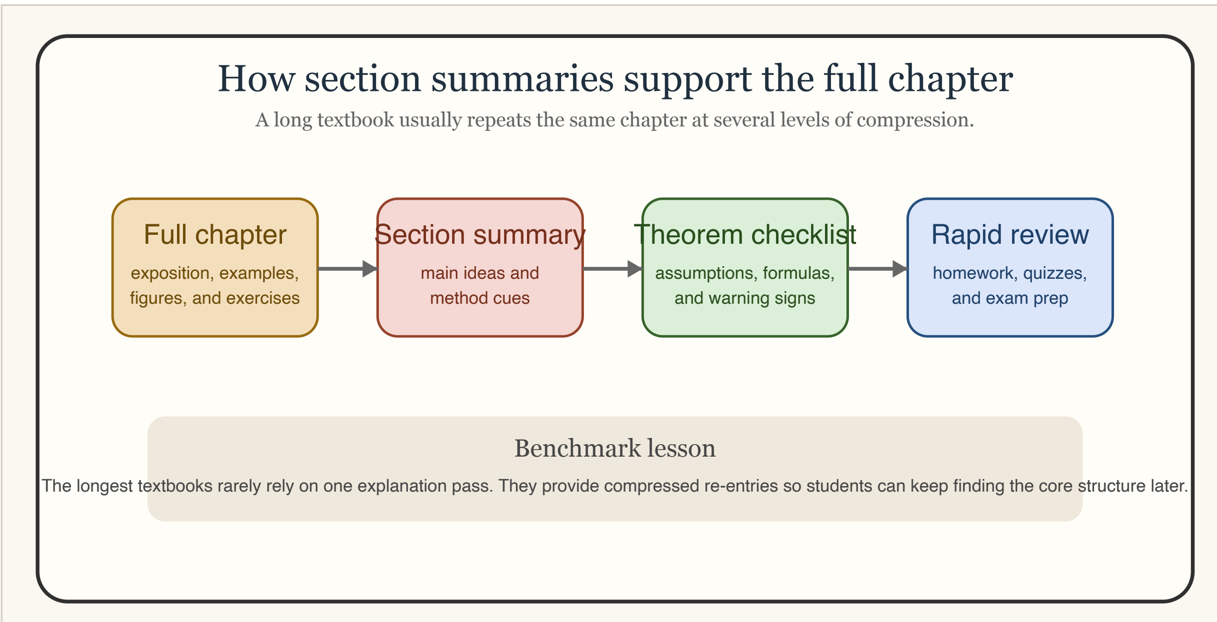
For a cumulative calculus exam, a workable short schedule is:

- Day 1: chapter review sheets only, marking weak sections.
- Day 2: oral prompts from Parts I-III and one page of derivative/integral drills.
- Day 3: oral prompts from Parts IV-VI and two worked examples rebuilt from memory.
- Day 4: mixed homework or quiz set under time pressure.
- Day 5: error-correction day using the answers, hints, and worked atlas.
- Day 6: one practice test and one oral pass through the prompts you still miss.
- Day 7: light review only: formulas, theorem hypotheses, method-choice cues, and sleep.

The key idea is retrieval with correction. Reading passively is useful early, but late-stage preparation improves more quickly when the student tries to produce the idea before looking.

Appendix AB. Section Summaries and Theorem Checklists

This appendix imitates one of the most useful density layers in large classroom texts: short section-level summaries that help students see the structure of a chapter before they get lost in the details. It is not meant to replace the full exposition. It is meant to help readers relocate the major claims, formulas, and decision cues quickly.



How to use this appendix

- Read a chapter summary before starting the chapter if you want a roadmap.
- Read it again after the chapter if you want to see what the main thread was.
- Use the theorem checklist before homework, quizzes, and exams to verify that assumptions are not being skipped.
- Use the "If you see this, think this" cues as method-selection prompts.

Chapter 1. Quantities, Functions, and the Shape of Change

`Sections at a glance`

- Varying quantities are the raw material of calculus.
- Functions can be read from formulas, tables, graphs, and words.
- Average rate of change belongs to an interval, not a point.
- Local linear thinking prepares the way for derivatives.

`Theorem and idea checklist`

- average rate = output change divided by input change,
- units belong in every interpretation sentence,
- local behavior may differ sharply from whole-interval behavior.

`If you see this, think this`

- table of values -> estimate average change first,
- contextual problem -> define inputs, outputs, and units before computing,
- "nearby" language -> local linearity or future derivative ideas are close.

Chapter 2. Nearness, Limits, and Continuity

`Sections at a glance`

- Limits describe nearby behavior.
- One-sided limits matter when rules or behavior change by direction.
- Continuity means value and nearby trend match.
- Algebra often reveals hidden holes or cancellations.

`Theorem and idea checklist`

- a two-sided limit exists only if both one-sided limits exist and agree,
- continuity at `a` requires $f(a)$ to exist and equal $\lim_{(x \to a)} f(x)$,
- quotient law needs nonzero denominator limit.

`If you see this, think this`

- $0/0$ -> simplify before substituting,
- piecewise rule -> check left-hand and right-hand behavior,
- hole on a graph -> the limit may still exist.

Chapter 3. Derivatives

`Sections at a glance`

- Derivatives are limits of secant slopes.

- The derivative is both slope and instantaneous rate of change.
- Differentiability is stronger than continuity.
- Higher derivatives track change of change.

`Theorem and idea checklist`

- derivative definition uses a limiting difference quotient,
- differentiability implies continuity,
- negative derivative means decreasing local behavior.

`If you see this, think this`

- motion context -> derivative means velocity,
- tangent line request -> compute a point and a slope,
- corner or cusp -> derivative may fail to exist.

Chapter 4. Working with Derivatives

`Sections at a glance`

- Derivative rules compress repeated limit work.
- The chain rule organizes nested dependence.
- Implicit differentiation keeps derivative logic available without solving for `y`.
- Linearization turns slope information into estimation.

`Theorem and idea checklist`

- product rule, quotient rule, and chain rule each answer a structural question,
- implicit differentiation always attaches `y'` to differentiated `y`-terms,
- a linearization is trustworthy only near the base point.

`If you see this, think this`

- expression inside another expression -> chain rule,
- `x` and `y` mixed in one equation -> implicit differentiation,
- estimation near an easy point -> linearization.

Chapter 5. What Derivatives Tell Us

`Sections at a glance`

- Sign of `f'` controls increase and decrease.
- Critical points are candidates for extrema.

- Sign of `f''` controls concavity.
- Optimization is context plus derivative evidence.

`Theorem and idea checklist`

- first-derivative test uses sign change, not only zero derivative,
- second-derivative test needs a critical point plus nearby second-derivative data,
- closed-interval extrema require endpoint checks.

`If you see this, think this`

- "maximum" or "minimum" -> define the objective and the domain first,
- graph shape from data -> sign chart and concavity,
- zero second derivative -> possible inflection, not automatic inflection.

Chapter 6. The Integral as Accumulation

`Sections at a glance`

- The definite integral is accumulated total.
- Signed area is one interpretation, not the only one.
- Riemann sums explain where the integral comes from.
- Average value is accumulated total divided by interval length.

`Theorem and idea checklist`

- definite integrals are numbers,
- negative integrand values contribute negative accumulation,
- units for a definite integral multiply quantity units by input units.

`If you see this, think this`

- rate in context -> integrate to recover total change,
- many tiny pieces -> Riemann-sum reasoning,
- average level over an interval -> average-value formula.

Chapter 7. Antiderivatives and the Fundamental Theorem

`Sections at a glance`

- Antiderivatives form families.
- Initial conditions choose one member of the family.

- The Fundamental Theorem links rates and accumulation.
- Substitution recognizes inside-derivative structure.

`Theorem and idea checklist`

- indefinite integrals need `+ C`,
- FTC Part I differentiates accumulation functions under continuity assumptions,
- FTC Part II evaluates definite integrals through any antiderivative.

`If you see this, think this`

- variable upper limit -> FTC Part I,
- definite integral and easy antiderivative -> FTC Part II,
- inside function with matching derivative nearby -> substitution.

Chapter 8. More Integration Tools

`Sections at a glance`

- Integration often begins with method choice.
- Integration by parts reverses the product rule.
- Trigonometric methods use structural identities.
- Partial fractions break rational expressions into integrable pieces.
- Numerical methods matter when exact antiderivatives are awkward or unavailable.

`Theorem and idea checklist`

- parts formula: $intudv = uv - intvdu$,
- partial fractions needs a factored denominator,
- numerical integration gives approximations, not exact symbolic identities.

`If you see this, think this`

- product with one factor simplifying on differentiation -> parts,
- rational function with lower-degree numerator -> partial fractions,
- data table or no clean antiderivative -> numerical method.

Chapter 9. Applications of Integration

`Sections at a glance`

- Area, volume, mass, work, and surface area all come from local pieces.
- A good setup begins with the geometry of one representative slice or shell.

- Units validate an application integrand.
- Different methods can describe the same object.

`Theorem and idea checklist`

- area between curves uses top minus bottom or right minus left,
- shell method needs radius and height,
- center of mass divides moment by total mass.

`If you see this, think this`

- rotation -> choose slices or shells before integrating,
- density function -> local mass = density times small geometry piece,
- work by variable force -> integrate force over displacement.

Chapter 10. Sequences and Series

`Sections at a glance`

- Sequences ask about terms; series ask about partial sums.
- Geometric behavior is the first benchmark.
- Convergence tests compare unknown behavior with known families.
- Taylor polynomials turn derivatives into approximation tools.

`Theorem and idea checklist`

- $a_n \to 0$ is necessary, not sufficient, for series convergence,
- ratio and root tests are strong for factorial and exponential structure,
- Taylor error estimates justify trust in approximation.

`If you see this, think this`

- positive-term comparison shape -> comparison or integral test,
- factorials or powers -> ratio or root test,
- local approximation near a center -> Taylor polynomial.

Chapter 11. Differential Equations and Models

`Sections at a glance`

- Differential equations encode rate laws.
- Slope fields show global behavior before exact solutions appear.
- Exponential and logistic models answer different growth questions.

- Numerical methods extend calculus beyond closed-form answers.

`Theorem and idea checklist`

- separable equations isolate variables before integrating,
- equilibria come from setting the derivative equal to zero,
- Euler's Method is repeated local linearization.

`If you see this, think this`

- rate proportional to amount -> exponential model,
- self-limiting growth -> logistic model,
- no easy formula -> use slope fields or numerical steps.

Chapter 12. Vectors and Space

`Sections at a glance`

- Points locate positions; vectors record displacement and direction.
- Lines use direction vectors; planes use normal vectors.
- Motion in space uses vector-valued functions.
- Dot and cross products encode geometry algebraically.

`Theorem and idea checklist`

- speed is the magnitude of velocity,
- projection measures the component in a chosen direction,
- cross-product magnitude gives parallelogram area.

`If you see this, think this`

- angle or alignment -> dot product,
- area and orientation -> cross product,
- tangent direction in space -> derivative of a vector function.

Chapter 13. Multivariable Functions

`Sections at a glance`

- Surfaces and contour maps show different views of one object.
- Limits in several variables are subtler because approach path matters.
- Partial derivatives isolate one-variable change within a multivariable setting.
- The gradient packages steepest increase information.

`Theorem and idea checklist`

- path testing can disprove a limit but cannot prove it,
- gradient is perpendicular to level sets,
- second-derivative test needs a critical point and suitable smoothness assumptions.

`If you see this, think this`

- contour map -> gradient and level-set geometry,
- local estimate near a point -> tangent plane or linearization,
- constrained optimization -> Lagrange multipliers.

Chapter 14. Multiple Integration

`Sections at a glance`

- Double and triple integrals accumulate over regions and solids.
- Iterated integrals organize a multi-dimensional total in stages.
- Coordinate choice can simplify geometry dramatically.
- Jacobian-style factors correct for geometric distortion.

`Theorem and idea checklist`

- polar coordinates use dA = rdrdtheta,
- cylindrical and spherical coordinates change both bounds and volume factors,
- reversing order is a geometric rewriting, not a symbolic trick.

`If you see this, think this`

- circular symmetry -> polar or cylindrical coordinates,
- ugly inner bounds -> consider reversing order,
- density over area or volume -> mass or probability total.

Chapter 15. Vector Calculus

`Sections at a glance`

- Vector fields assign directional data to points in space.
- Line integrals measure circulation or work along paths.
- Surface integrals measure flux through oriented surfaces.
- Green, Divergence, and Stokes convert local behavior into boundary totals.

`Theorem and idea checklist`

- conservative fields admit potentials on suitable domains,
- orientation changes the sign of circulation or flux,
- theorem choice depends on the geometry of the boundary and the quantity requested.

`If you see this, think this`

- closed-loop work in a conservative field -> maybe zero,
- outward flow across a closed surface -> Divergence Theorem,
- circulation around a boundary curve -> Green or Stokes depending on dimension.

Chapter 16. Parametric Curves, Polar Coordinates, and Curvature

`Sections at a glance`

- Parametrization tracks order, direction, and repeated tracing.
- Polar coordinates recast planar geometry in terms of radius and angle.
- Arc length and polar area grow from local geometric pieces.
- Curvature measures turning, not merely slope.

`Theorem and idea checklist`

- $dy/dx = (dy/dt)/(dx/dt)$ when $dx/dt \neq 0$,
- polar area uses $(1/2) intr^2 dtheta$,
- curvature depends on rate of change of the tangent direction.

`If you see this, think this`

- traced motion with timing information -> keep the parameter,
- origin-centered geometry -> think polar,
- question about turning sharpness -> curvature rather than slope.

Chapter 17. Second-Order Differential Equations and Oscillation

`Sections at a glance`

- Second-order equations encode acceleration and restoring-force models.
- Characteristic roots classify solution families.
- Damping, forcing, and resonance describe qualitatively different regimes.
- Boundary-value problems ask for shape conditions rather than initial motion data.

`Theorem and idea checklist`

- repeated roots need an extra factor of `t`,
- complex roots generate sine and cosine behavior,
- two initial conditions are typically required for uniqueness in a second-order IVP.

`If you see this, think this`

- oscillation around equilibrium -> complex roots or sinusoidal terms,
- resistance or energy loss -> damping,
- external periodic input -> forced response and possible resonance.

Chapter 18. Improper Integrals and Long-Tail Behavior

`Sections at a glance`

- Improper integrals are limits of proper integrals.
- Convergence depends on tail behavior or singular behavior.
- Comparison with benchmark functions is the main strategy.
- Improper integrals link continuous accumulation to series and probability.

`Theorem and idea checklist`

- write the limit before evaluating,
- split the integral at every singular point,
- compare with $1/x^p$ behavior whenever possible.

`If you see this, think this`

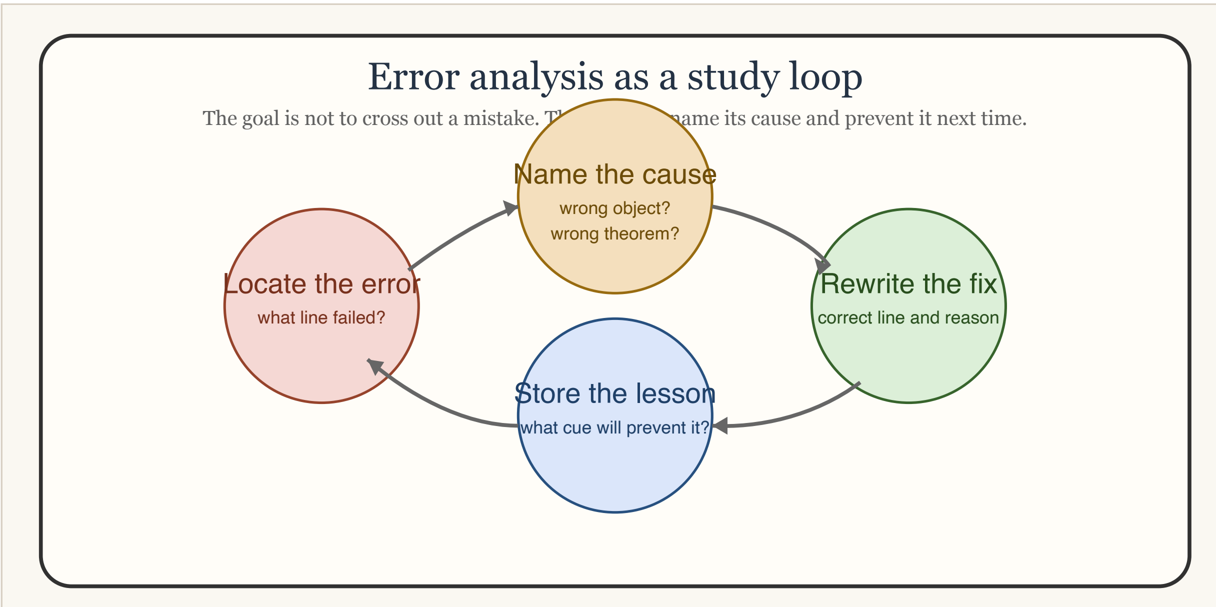
- infinite interval -> tail comparison,
- blow-up at an endpoint -> local singularity comparison,
- positive decreasing series and related integral -> integral test.

Closing note

Long textbooks earn many pages by restating the course at several levels of compression: full exposition, worked examples, section summaries, chapter reviews, and assessment sets. This appendix is one more layer in that system. Its purpose is not repetition for its own sake. Its purpose is retrieval with structure.

Appendix AC. Error Analysis Casebook

Long commercial texts often devote many pages to worked pitfalls, sample mistakes, and "what went wrong?" prompts. That density is not accidental. Students do not only need more correct examples. They also need explicit practice diagnosing plausible wrong moves.



How to use this appendix

- Read each broken solution before reading the repair.
- State the error in words, not only by crossing something out.
- Ask what false idea made the wrong step seem reasonable.
- Rewrite the fix in full, not as a one-symbol correction.

Chapter 1. Functions and change

Error 1

Broken claim: "The quantity changed by `6`, so the rate is `6`."

Repair: a change is not a rate until it is divided by the input change. A rate must compare two changes and should include units.

Error 2

Broken claim: "Because two graphs have the same slope at one point, they have the same average rate on every interval."

Repair: local slope information and average interval behavior answer different questions.

Chapter 2. Limits and continuity

Error 1

Broken line: $f(2) = 5$, so $\lim_{(} x \to 2) f(x) = 5$.

Repair: a function value and a limit can agree, but one does not force the other. Nearby behavior must still be checked.

Error 2

Broken line: "The left side and right side both look close in a table, so the limit is proved."

Repair: a table gives evidence, not proof. Algebra or structure must justify the pattern.

Error 3

Broken line: $\lim_{(} x \to 2)(x^2 - 4)/(x - 2) = (4 - 4)/(2 - 2) = 0$.

Repair: $0/0$ is not a value. It is a signal to simplify before evaluating.

Chapter 3. Derivatives

Error 1

Broken line: "The derivative at $x = 3$ is the same as the derivative function."

Repair: the derivative function assigns a slope to each input. Evaluating it at one point gives one number from that whole function.

Error 2

Broken line: "A derivative of `0` means the graph is flat, so it must be a max or min."

Repair: derivative zero gives a candidate, not a conclusion. The sign on either side still matters.

Error 3

Broken line: "If the graph is continuous, it has a derivative everywhere."

Repair: corners, cusps, and vertical tangents show otherwise.

Chapter 4. Working with derivatives

Error 1

Broken line: $d/dx[(x^2 + 1)^4] = 4(x^2 + 1)^3$.

Repair: the inner derivative `2x` is missing. This is a chain-rule failure.

Error 2

Broken line: differentiating $x^2 + y^2 = 9$ gives $2x + 2y = 0$.

Repair: `y` depends on `x`, so the derivative of y^2 is `2y y'`.

Error 3

Broken claim: "The linearization gave a value, so it must be accurate."

Repair: a linear approximation is local. Its quality depends on distance from the base point and graph curvature.

Chapter 5. What derivatives tell us

Error 1

Broken line: "$f''(c) = 0$, so `c` is an inflection point."

Repair: second derivative zero only marks a candidate. Concavity must actually change.

Error 2

Broken line: solving $f'(x) = 0$ and reporting $x = 4$ as the final optimization answer.

Repair: optimization ends in context, not in an unlabeled variable value. Return to the original quantity and units.

Error 3

Broken claim: "Since $f'(x) > 0$, the graph is concave up."

Repair: positive first derivative means increasing; concavity depends on the second derivative.

Chapter 6. The integral as accumulation

Error 1

Broken claim: "A negative definite integral must be wrong because area cannot be negative."

Repair: a definite integral can represent signed accumulation, not only geometric area.

Error 2

Broken line: using $\int_a^b f(x)dx$ when the context asks for total distance and the rate changes sign.

Repair: total distance uses $|v(t)|$, not raw signed velocity.

Error 3

Broken claim: "A definite integral is just an antiderivative with limits."

Repair: the object type matters. A definite integral is a number representing a total; an antiderivative is a function family.

Chapter 7. Antiderivatives and the Fundamental Theorem

Error 1

Broken line: $int 6x dx = 3x^2$.

Repair: indefinite integrals require `+ C`.

Error 2

Broken line: $\int_0^2 6x dx = 3x^2 + C|_0^2$.

Repair: definite integrals do not need `+ C`. The constant cancels in any antiderivative difference.

Error 3

Broken claim: "Substitution is just changing letters."

Repair: substitution succeeds only when the new variable compresses the structure and the differential relationship is accounted for.

Chapter 8. More integration tools

Error 1

Broken line: choosing parts for $int 2x \cos(x^2) dx$.

Repair: substitution is the natural choice because the inside derivative is present.

Error 2

Broken line: partial fractions on a denominator that was never factored.

Repair: factor first. The decomposition depends on the denominator's algebraic structure.

Error 3

Broken claim: "The trapezoidal rule gave `3.1`, so the exact answer is `3.1`."

Repair: numerical integration produces an approximation and should be labeled as such.

Chapter 9. Applications of integration

Error 1

Broken line: area between curves written as $int(\text{lower} - \text{upper})$.

Repair: the sign tells you the setup is wrong for a positive area problem.

Error 2

Broken line: shell method radius taken from the wrong axis.

Repair: radius is distance from the slice to the axis of rotation, not distance to the graph.

Error 3

Broken claim: "The answer `18` is complete."

Repair: application answers need units and often need interpretation in context.

Chapter 10. Sequences and series

Error 1

Broken claim: "$a_n \to 0$, so $\mathrm{sum}\, a_n$ converges."

Repair: term limits are necessary but not sufficient.

Error 2

Broken line: applying the ratio test and stopping before checking whether the ratio limit is less than, greater than, or equal to `1`.

Repair: the ratio test conclusion depends entirely on that comparison.

Error 3

Broken line: using a Taylor polynomial far from its center without mentioning reliability.

Repair: approximation quality depends on distance from the center and remainder control.

Chapter 11. Differential equations and models

Error 1

Broken claim: "Since the equation has a derivative, the solution must be linear."

Repair: every differential equation involves derivatives; the solution family depends on the structure of the rate law.

Error 2

Broken line: Euler's Method using the slope from the wrong point.

Repair: each step uses the slope at the current point, then advances.

Error 3

Broken claim: "Logistic growth is just exponential growth with a bigger formula."

Repair: the limiting factor changes the long-term behavior qualitatively by building in self-limitation.

Chapter 12. Vectors and space

Error 1

Broken line: subtracting points in the wrong order and then interpreting the vector backward.

Repair: displacement from `A` to `B` is `B - A`, and direction matters.

Error 2

Broken claim: "Speed is the same as velocity."

Repair: speed is magnitude only; velocity includes direction.

Error 3

Broken line: using the dot product formula when the question asks for area.

Repair: area from two vectors points toward the cross product.

Chapter 13. Multivariable functions

Error 1

Broken claim: "Two tested paths matched, so the limit exists."

Repair: matching paths do not prove existence; mismatching paths disprove it.

Error 2

Broken line: taking a directional derivative with an unnormalized direction vector.

Repair: normalize first so the answer measures change per unit step.

Error 3

Broken claim: "The gradient is just a storage device for partial derivatives."

Repair: the gradient also carries geometric meaning: steepest increase direction and level-set normal.

Chapter 14. Multiple integration

Error 1

Broken line: polar double integral written without the factor `r`.

Repair: coordinate change stretches area elements; `r` is the correction factor.

Error 2

Broken line: iterated bounds that do not describe the stated region.

Repair: sketch the region first, then read bounds from the sketch.

Error 3

Broken claim: "Changing order of integration changes the value."

Repair: if both iterated forms describe the same region and integrand correctly, the value is the same.

Chapter 15. Vector calculus

Error 1

Broken claim: "A line integral always measures flux."

Repair: `F . dr` measures work or circulation along a path; flux is across a boundary with a normal.

Error 2

Broken line: using the Divergence Theorem on an open surface.

Repair: the theorem requires a closed boundary surface enclosing a volume.

Error 3

Broken claim: "Orientation changes the picture but not the answer."

Repair: reversing orientation changes the sign of circulation or flux.

Chapter 16. Parametric curves, polar coordinates, and curvature

Error 1

Broken line: setting $dy/dx = dy/dt$.

Repair: the slope in the plane is $(dy/dt)/(dx/dt)$ when $dx/dt \neq 0$.

Error 2

Broken claim: "A polar point has one unique representation."

Repair: adding 2pi to the angle or changing sign with a shifted angle can represent the same point.

Error 3

Broken claim: "Curvature is just another word for slope."

Repair: curvature measures turning, not tilt.

Chapter 17. Second-order differential equations and oscillation

Error 1

Broken line: repeated-root solution written as $C_1e^(rt) + C_2e^(rt)$.

Repair: the second term must be multiplied by `t`.

Error 2

Broken claim: "Complex roots mean the solution is not real."

Repair: complex roots combine into real sine-cosine solution families.

Error 3

Broken claim: "Damping changes only the amplitude, so it does not affect the character of the system."

Repair: damping can alter whether oscillations persist at all.

Chapter 18. Improper integrals and long-tail behavior

Error 1

Broken line: computing an improper integral without writing the limit.

Repair: convergence is part of the problem, so the limiting process must be explicit.

Error 2

Broken claim: "Since the integrand goes to zero, the integral converges."

Repair: tail decay can still be too slow, as `1/x` shows.

Error 3

Broken line: comparing in the wrong direction.

Repair: to prove convergence, dominate the integrand by a known convergent benchmark; to prove divergence, dominate a known divergent benchmark from below.

Closing advice

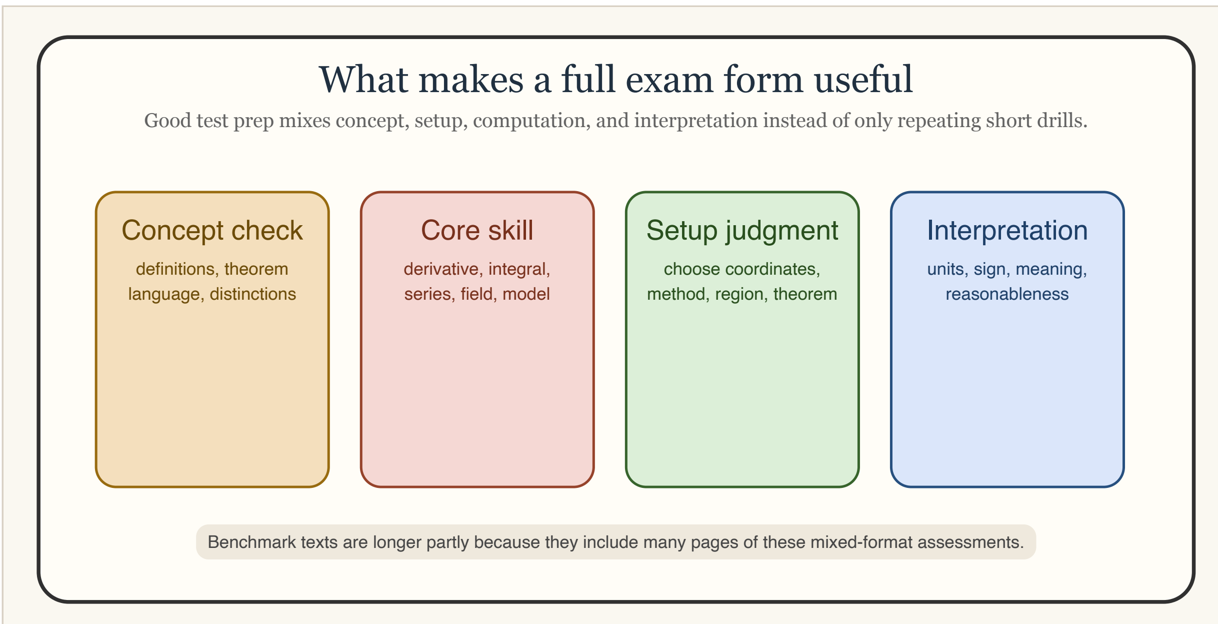
Students often think a mistake is only a sign error or a dropped factor. In calculus, many of the most expensive errors happen earlier:

- choosing the wrong object,
- choosing the wrong method,
- forgetting a theorem hypothesis,
- or answering the wrong question.

That is why large textbooks spend so much space on diagnostic infrastructure. A reader who can detect these patterns early needs fewer rescue steps later.

Appendix AE. Full-Length Exam Forms and Solution Outlines

Large textbooks often include long review sets, chapter tests, and practice finals that are close to real classroom assessments. This appendix adds that density layer in a more serious format than the short quiz banks.



How to use these forms

- Work one form under time pressure before consulting the outline.
- Mark whether each missed item was a concept issue, setup issue, or algebra issue.
- Rework the missed problems from memory the next day.
- Pair these forms with the board-style oral prompts and the worked-solution atlas for deeper review.

Form A. Calculus I Comprehensive Exam

1. Estimate $\lim_{x \to 2}(x^2 + 3x - 1)$ and then compute it exactly.
2. Compute $\lim_{x \to 2}(x^2 - 4)/(x - 2)$.
3. Explain why continuity does not imply differentiability.
4. Use the limit definition to find the derivative of $f(x) = x^2$ at a general `x`.
5. Differentiate $g(x) = (3x^2 + 1)^5$.

6. Differentiate $h(x) = x^2 e^x$.
7. Differentiate the implicit relation $x^2 + y^2 = 25$.
8. Find the tangent line to $f(x) = \sqrt{x}$ at $x = 4$.
9. Find and classify the critical points of $p(x) = x^3 - 3x$.
10. Determine intervals of concavity for $q(x) = x^4 - 4x^2$.
11. A rectangle has perimeter `80`. Find the dimensions of greatest area.
12. Interpret and evaluate $\int_0^3 (4 - x)dx$.
13. Use the Fundamental Theorem to differentiate $F(x) = \int_1^x (t^2 + 3)dt$.
14. Compute $\int_0^1 2x(x^2 + 1)^3 dx$.
15. Explain the difference between net change and total distance traveled.

`Outline answers`

1. `9`
2. `4`
3. `2x`
4. $30x(3x^2 + 1)^4$
5. $e^x(2x + x^2)$
6. $y' = -x/y$
7. $y = 1/4x + 1$
8. local max at $x = -1$, local min at $x = 1$
9. concave down on $(-1/\sqrt{3}, 1/\sqrt{3})$, concave up outside
10. `20` by `20`
11. $15/2$, interpreted as accumulated total
12. $x^2 + 3$
13. $15/4$

Form B. Calculus II Comprehensive Exam

1. Find an antiderivative of $6x^2 - 4x + 1$.
2. Solve the initial-value problem $y' = 4x, y(0) = 2$.
3. Compute $int \times e^x dx$.
4. Compute $\int_0^1 2x \cos(x^2)dx$.
5. Decompose $(3x + 5)/(x^2 + x - 2)$ and integrate.
6. Estimate $\int_0^2 x^2 dx$ by the midpoint rule with $n = 2$.
7. Find the area between $y = x$ and $y = x^2$ on `[0, 1]`.
8. Set up the shell-method volume for rotating the region under $y = x$ on `[0, 1]` about the `y`-axis.
9. Find the average value of $f(x) = x^2 + 1$ on `[0, 2]`.
10. Determine whether $sum 1/n^2$ converges.
11. Determine whether $sum 1/n$ converges.

12. Use the ratio test on $sum n!/10^n$.
13. Write the quadratic Maclaurin polynomial for e^x.
14. Use the cubic Maclaurin polynomial for sin x to approximate $\sin(0.4)$.
15. Explain why $a_n \to 0$ does not guarantee that $sum a_n$ converges.

`Outline answers`

1. $2x^3 - 2x^2 + x + C$
2. $y = 2x^2 + 2$
3. $e^x(x - 1) + C$
4. sin 1
5. $\ln|x + 2| + 2\ln|x - 1| + C$
6. `2.5`
7. $1/6$
8. $7/3$
9. converges
10. diverges
11. diverges
12. $1 + x + x^2/2$
13. about `0.38933`

Form C. Differential Equations and Series Synthesis Exam

1. Explain the difference between a sequence and a series.
2. Find the sum of $3 + 3/2 + 3/4 + \ldots$.
3. State the interval of convergence of $sum n((x - 2)^n)/3^n$.
4. Use an alternating-series remainder idea to estimate $1 - 1/3 + 1/5 - 1/7$.
5. Solve $dy/dx = ky$ with $y(0) = 5$.
6. Analyze the equilibria of $dP/dt = 0.4P(1 - P/500)$.
7. Use Euler's Method with $h = 0.1$ to estimate $y(0.2)$ for $y' = x + y, y(0) = 1$.
8. Solve $y'' + 4y = 0$ with $y(0) = 3, y'(0) = -2$.
9. Solve $y'' + 2y' + 10y = 0$ with $y(0) = 1, y'(0) = 0$.
10. Explain the difference between underdamped and overdamped motion.
11. Describe how Taylor polynomials and Euler's Method both use local information.
12. Determine whether $\int_1^i nfinity 1/x^2 dx$ converges.
13. Determine whether $\int_1^i nfinity 1/x dx$ converges.
14. Use comparison to classify $\int_1^i nfinity 1/(x^2 + 4x + 3)dx$.
15. Explain how the integral test connects series to improper integrals.

`Outline answers`

1. `6`
2. `(−1, 5)`
3. estimate 76/ 105, error at most 1/ 9
4. $y = 5e^{(}kx)$
5. equilibria at `0` and `500`, with `500` stable
6. `1.22`
7. $3\cos 2t - \sin 2t$
8. $e^{(} - t)(\cos 3t + (1/3)\sin 3t)$
9. converges to `1`
10. diverges
11. converges by comparison with $1/x^2$

Form D. Calculus III and Vector Calculus Comprehensive Exam

1. Find the angle between $\langle 1,2,2\rangle$ and $\langle 2,1,2\rangle$.
2. Write the equation of the plane through `(1, -2, 4)` with normal $\langle 2,-1,3\rangle$.
3. Find the tangent line to $r(t) = \langle t,t^2,t^3\rangle$ at $t = 1$.
4. Compute ∇f for $f(x,y) = x^2y + y^2$ at `(1, 2)`.
5. Find the directional derivative of that function at `(1, 2)` in the direction $\langle 3,4\rangle$.
6. Classify the critical point of $f(x,y) = x^2 + y^2 - 6x + 2y$.
7. Reverse the order of $\int_0^1 \int_x^1 f(x,y)dydx$.
8. Evaluate $\iint_R (x^2 + y^2)dA$ over the disk $x^2 + y^2 \le 4$.
9. Find the maximum and minimum of $f(x,y) = xy$ on $x^2 + y^2 = 1$.
10. Compute the work of $F(x,y) = \langle y,x\rangle$ along the segment from `(0, 0)` to `(1, 2)`.
11. Use Green's Theorem on $F(x,y) = \langle -y,x\rangle$ around the unit circle.
12. Compute the outward flux of $F(x,y,z) = \langle x,y,z\rangle$ across the sphere of radius `2`.
13. Use Stokes' Theorem on $F(x,y,z) = \langle -y,x,0\rangle$ for the unit circle in the plane $z = 0$.
14. Explain the difference between flux and circulation.
15. Explain why Green, Divergence, and Stokes belong to one local-to-global family.

`Outline answers`

1. $\arccos(8/9)$
2. $2x - y + 3z - 16 = 0$
3. $\langle 1,1,1\rangle + s\langle 1,2,3\rangle$
4. $\langle 4,5\rangle$
5. 32/ 5
6. local minimum at `(3, -1)`
7. $\int_0^1 \int_0^y f(x,y)dxdy$
8. 8pi

9. max 1/ 2, min – 1/ 2
10. `2`
11. 2pi
12. 32pi
13. 2pi

Form E. Full Sequence Final

1. Distinguish a limit from a function value.
2. Differentiate $f(x) = (x^2 + 1)^4$.
3. Find the local extrema of $x^3 - 3x$.
4. Compute $\int_0^1 2xe^(x^2)dx$.
5. Set up the volume of rotating $y = \sqrt{x}$ on `[0, 4]` about the `x`-axis.
6. Explain why the harmonic series is a standard counterexample.
7. Solve $dy/dx = 3y$ with $y(0) = 2$.
8. Compute f_x and f_y for $f(x,y) = x^2 + xy$.
9. Explain why a multivariable path test can disprove but not prove a limit.
10. Find the slope of $x = t^2 + 1, y = t^3 - t$ at $t = 1$.
11. Determine whether $\int_1^i nfinity 1/x^2 dx$ converges.
12. Explain what the gradient generalizes from one-variable calculus.
13. Explain the difference between a line integral and a surface integral.
14. Describe one setting where numerical methods are more realistic than exact symbolic answers.
15. Compare the Fundamental Theorem of Calculus with one higher-dimensional theorem in a short paragraph.

`Outline answers`

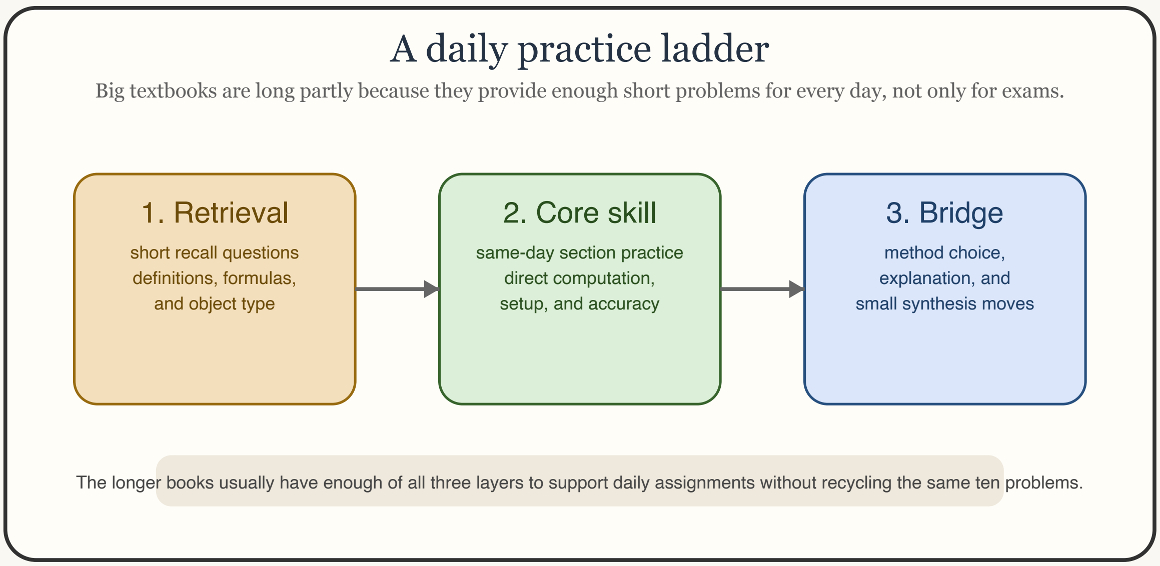
1. $8x(x^2 + 1)^3$
2. local max at $x = -1$, local min at $x = 1$
3. `e - 1`
4. $2e^(3x)$
5. $f_x = 2x + y, f_y = x$
6. `1`
7. yes, value `1`

Instructor note

Texts in the `1,100`-plus-page range usually feel long partly because they contain many pages of test forms, chapter exams, and review packets in addition to the explanatory chapters themselves. This appendix intentionally adds that layer so the manuscript functions more like a course system and less like a compact set of notes.

Appendix AF. Daily Review and Skill Builder Sets

This appendix is built for sheer classroom usability. Large calculus texts are long partly because they contain enough short assignable problems for daily retrieval, recitation warm-ups, and low-stakes skill checks. The sets below are intentionally compact, direct, and easy to split across a week.



How to use these sets

- Use problems 1 – 3 as recall warm-ups.
- Use problems 4 – 7 as same-day section practice.
- Use problems 8 – 10 as bridge problems that force explanation or choice of method.

Chapter 1 set

1. A quantity rises from `15` to `33` in `6` minutes. Find the average rate of change.
2. State the units in problem 1.
3. Write one sentence distinguishing a quantity from a rate.
4. Interpret the slope in $C(m) = 5 + 1.9m$.
5. Give an example of a situation where a table is a natural representation of a function.
6. Give an example of a situation where a graph is more informative than a formula.

7. Explain why average rate of change belongs to an interval.
8. Sketch two functions with the same average rate on `[0, 2]`.
9. Describe a setting in which local linear thinking would help estimation.
10. Explain why units belong in the final sentence of a rate problem.

Chapter 2 set

1. Compute $\lim_{(x \to 1)}(x^2 + 3x)$.
2. Compute $\lim_{(x \to 2)}(x^2 - 4)/(x - 2)$.
3. State the difference between a left-hand and a right-hand limit.
4. Give an example of a removable discontinuity.
5. Give an example of a jump discontinuity.
6. Explain why $f(a)$ and $\lim_{(x \to a)} f(x)$ are different objects.
7. Describe one situation where a table suggests a limit but does not prove it.
8. Explain why continuity requires agreement between value and nearby behavior.
9. State one condition that can cause the quotient law for limits to fail.
10. Sketch a graph with a hole at $x = 3$ but a two-sided limit there.

Chapter 3 set

1. State the derivative definition.
2. Use the known rule to compute the derivative of x^3 at $x = 2$.
3. Interpret a negative derivative in words.
4. Interpret a second derivative in a motion setting.
5. Give an example of a function that is continuous but not differentiable at a point.
6. Explain why a derivative is usually a function rather than only a number.
7. Distinguish a secant slope from a tangent slope.
8. Sketch a graph with positive derivative and negative second derivative at a point.
9. Describe what the derivative of position measures.
10. Explain why instantaneous change is built from interval averages.

Chapter 4 set

1. Differentiate $x^6 - 4x + 3$.
2. Differentiate $(2x + 1)^5$.
3. Differentiate x^3e^x.
4. Differentiate $\ln(x^2 + 9)$.
5. Use implicit differentiation on $x^2 + xy + y^2 = 7$.
6. Find the linearization of $\sqrt{x}$ at $x = 9$.

7. Explain why the chain rule is about nested dependence.
8. Explain why linearization becomes weaker far from the base point.
9. Describe one signal that implicit differentiation is the right tool.
10. Compare exact evaluation and linear approximation in one short paragraph.

Chapter 5 set

1. Find the critical points of $x^3 - 3x$.
2. Determine where $x^3 - 3x$ is increasing.
3. Determine where $x^3 - 3x$ is decreasing.
4. Explain what concave up means geometrically.
5. Find `f''(x)` for $f(x) = x^4 - 4x^2$.
6. Explain why $f'(c) = 0$ is not enough to prove an extremum.
7. Describe what an inflection point means in words.
8. State the first-derivative test in plain language.
9. Explain why endpoint checks matter in a closed-interval optimization problem.
10. Describe one real context where the optimized quantity is not the most obvious quantity in the problem.

Chapter 6 set

1. Compute $\int_0^2 4dx$.
2. Compute $\int_0^3 x\, dx$.
3. Explain the difference between signed area and total area.
4. Write a left-endpoint Riemann sum for `[0, 4]` with `4` equal subintervals.
5. State what a definite integral returns.
6. Explain what units a definite integral has in a rate problem.
7. Describe why negative integrand values matter.
8. Explain how a Riemann sum becomes a definite integral conceptually.
9. State the average-value formula on `[a, b]`.
10. Describe one context where an integral represents net change rather than geometric area.

Chapter 7 set

1. Find an antiderivative of `8x`.
2. Solve $F'(x) = 6x, F(0) = 1$.
3. Evaluate $\int_0^1 2x\, dx$.
4. Differentiate $F(x) = \int_2^x (t^2 + 1)dt$.

5. Evaluate $\int_0^1 2x(x^2 + 1)^2 dx$.
6. Explain why antiderivatives differ by constants.
7. State the Fundamental Theorem of Calculus in words.
8. Explain why substitution works in one paragraph.
9. Describe what an accumulation function measures.
10. Distinguish a definite integral from an indefinite integral.

Chapter 8 set

1. Evaluate $int \times e^x dx$.
2. Evaluate $int \cos \times dx$.
3. Explain when substitution is preferable to parts.
4. Explain when parts is preferable to substitution.
5. Set up partial fractions for $1/(x^2 - 1)$.
6. Compute one trapezoidal estimate for $\int_0^2 f(x)dx$ using $f(0) = 1, f(1) = 2, f(2) = 5$.
7. Describe one cue for trigonometric substitution.
8. Explain why method choice is the hard part of Chapter 8.
9. Describe one setting where a numerical answer is better than a symbolic one.
10. Compare substitution and parts as two different ways of reading integrand structure.

Chapter 9 set

1. Set up the area between $y = x$ and $y = x^2$ on `[0, 1]`.
2. State the washer formula in words.
3. State the shell formula in words.
4. Write the arc-length formula for $y = f(x)$.
5. Write the mass integral for a rod with density $\varrho(x)$.
6. Explain why units can diagnose a flawed application setup.
7. Describe one signal that the shell method is easier than washers.
8. Explain why a work integral is an accumulation of local force contributions.
9. Compare mass and center-of-mass problems.
10. Explain why one representative slice is the key picture in many applications.

Chapter 10 set

1. Determine whether $a_n = 1/n$ converges.
2. Determine whether $a_n = (-1)^n$ converges.
3. State when a geometric series converges.
4. Find the sum of $2 + 1 + 1/2 + \ldots$.

5. Explain why $a_n \to 0$ is necessary for $suma_n$ to converge.
6. State one good use of the ratio test.
7. State one good use of the integral test.
8. Write the quadratic Maclaurin polynomial for e^x.
9. Explain why Taylor polynomials are local approximations.
10. Compare absolute and conditional convergence in words.

Chapter 11 set

1. Explain what a slope field shows.
2. Solve $dy/dx = 3x$.
3. Solve $dy/dx = 2y$.
4. Identify the equilibria of $y' = y(3 - y)$.
5. Explain what stable equilibrium means.
6. Perform one Euler step for $y' = x + y, y(0) = 1, h = 0.1$.
7. Compare exponential and logistic growth.
8. Explain the difference between model error and numerical error.
9. Describe why phase lines are useful.
10. Explain why Euler's Method is a local-linear method.

Chapter 12 set

1. Find the length of $\langle 2, 3, 6 \rangle$.
2. Find the dot product of $\langle 1, 2, 3 \rangle$ and $\langle 4, 0, -1 \rangle$.
3. Find the cross product of $\langle 1, 0, 0 \rangle$ and $\langle 0, 1, 0 \rangle$.
4. Write a vector equation for a line through `(1, 2, 3)` with direction $\langle 2, -1, 4 \rangle$.
5. Write the equation of a plane through `(1, -1, 2)` with normal $\langle 2, 3, -1 \rangle$.
6. Differentiate $r(t) = \langle t, t^2, t^3 \rangle$.
7. Explain the difference between speed and velocity.
8. Describe what projection means geometrically.
9. Explain why a cross product carries orientation.
10. Compare line equations and plane equations in one short paragraph.

Chapter 13 set

1. Compute f_x and f_y for $f(x, y) = x^2y + y^2$.
2. Find ∇f for $f(x, y) = x^2 + y^2$.
3. Explain what a contour line represents.
4. Explain why path testing can disprove a multivariable limit.

5. Explain why it cannot prove one.
6. State the tangent-plane approximation formula.
7. Explain why the gradient is perpendicular to level curves.
8. Describe a saddle point in words.
9. State the main idea of Lagrange multipliers.
10. Compare a tangent line in one variable with a tangent plane in two variables.

Chapter 14 set

1. Write a double integral over $0 \leq x \leq 2, 0 \leq y \leq 3$.
2. Explain what a triple integral accumulates.
3. State `dA` in polar coordinates.
4. Explain why the factor `r` appears.
5. Describe when changing order of integration helps.
6. Describe one reason to use cylindrical coordinates.
7. Describe one reason to use spherical coordinates.
8. Explain the geometric meaning of a Jacobian-like factor.
9. Compare mass density and probability density in an integral.
10. Explain why a region sketch usually comes before the bounds.

Chapter 15 set

1. Define a vector field.
2. Explain what a line integral of work measures.
3. Explain what flux measures.
4. State one sign that a field may be conservative.
5. Explain why orientation matters.
6. State Green's Theorem in words.
7. State the Divergence Theorem in words.
8. State Stokes' Theorem in words.
9. Compare circulation and flux.
10. Explain why the major vector-calculus theorems resemble the Fundamental Theorem of Calculus.

Chapter 16 set

1. Give a parametrization of the unit circle.
2. Compute dy/dx for $x = t^2 + 1, y = t^3 - t$.
3. Convert $(r, \theta) = (2, \pi/3)$ to rectangular coordinates.

4. State the polar-area formula.
5. State the parametric arc-length formula.
6. Explain why polar coordinates are not unique.
7. Explain why a parametrized curve may fail the vertical-line test.
8. Describe what curvature measures.
9. Compare rectangular and polar descriptions of the same curve.
10. Explain why the parameter can encode direction and speed.

Chapter 17 set

1. State why a second-order equation usually needs two initial conditions.
2. Write the characteristic equation for $y'' - 3y' + 2y = 0$.
3. Explain what repeated roots do to the solution form.
4. Explain why complex roots lead to oscillation.
5. Describe the effect of damping.
6. Describe the effect of forcing.
7. Explain resonance in words.
8. Contrast initial-value and boundary-value problems.
9. Rewrite $y'' = -4y - 0.5y'$ as a first-order system.
10. Compare underdamped and overdamped behavior.

Chapter 18 set

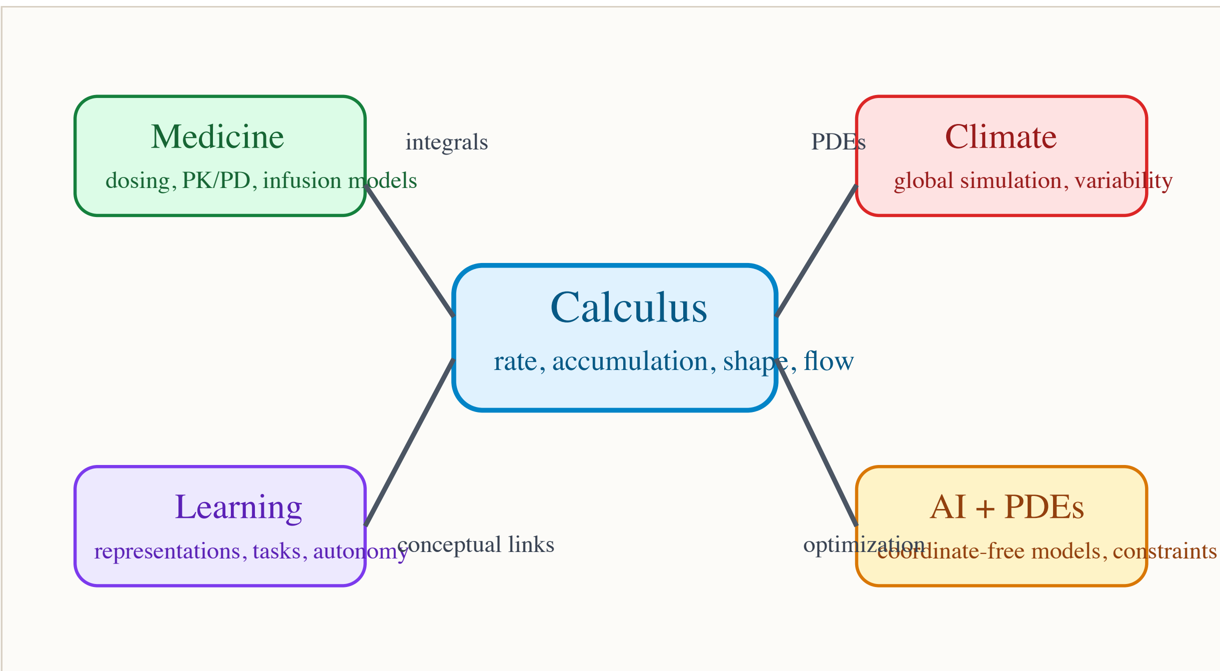
1. Explain why $\int_1^i nfinity 1/x^2 dx$ is improper.
2. State the limit definition for $\int_a^i nfinity f(x) dx$.
3. Determine whether $\int_1^i nfinity 1/x dx$ converges.
4. Determine whether $\int_0^1 1/\sqrt{x} dx$ converges.
5. State the `p`-integral threshold on `[1, infinity)`.
6. Explain why a function going to `0` is not enough for convergence.
7. Describe how the integral test links series and integrals.
8. Describe one probability interpretation of an improper integral.
9. Explain how comparison reasoning works for improper integrals.
10. Compare divergence caused by a tail with divergence caused by a singular endpoint.

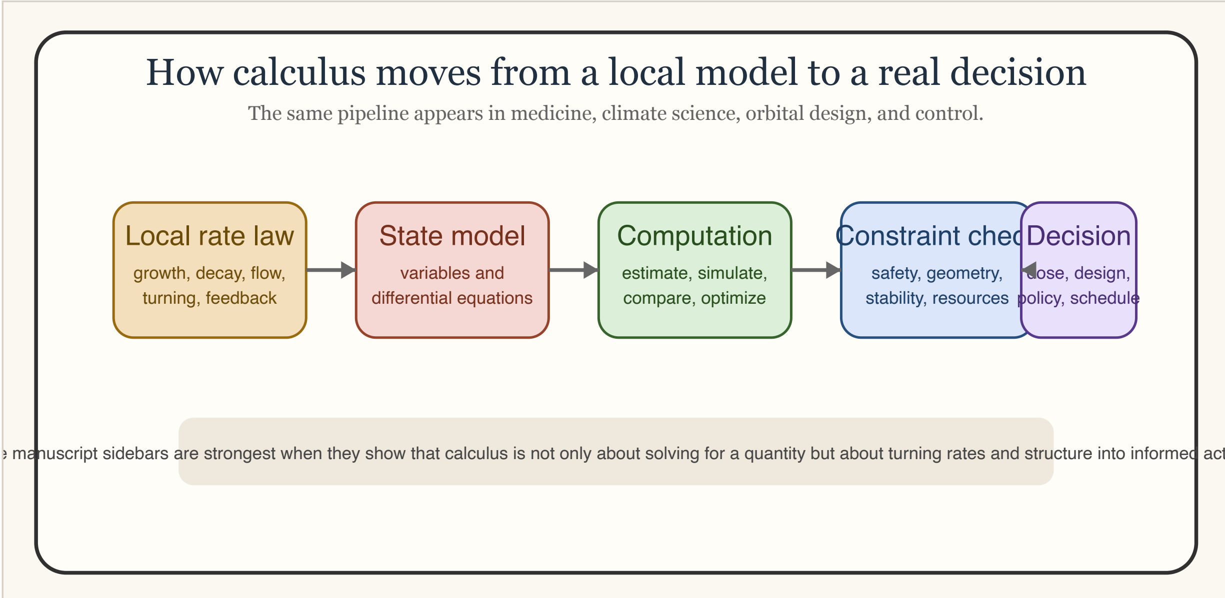
Appendix D. Research and Application Excerpts

These short pages are not part of the main calculus sequence. They are here to answer a different question:

Where does calculus show up now, in current teaching research and current scientific work?

The selections below were gathered in a scholar-style search pass on 2026-03-20 and then verified against primary-source pages. They are deliberately short. Each excerpt is meant to be a doorway, not a literature review.





Excerpt 1. Derivative understanding improves when representations are linked

Source: Farzad Radmehr and Melih Turgut, *Learning more about derivative: leveraging online resources for varied realizations*, ZDM, published March 28, 2024.
Link: https://link.springer.com/article/10.1007/s11858-024-01564-0

This paper studies how derivative ideas are presented in a highly viewed online resource and argues that students learn more effectively when derivative is not trapped in one form. Derivative-as-slope, derivative-as-rate, derivative-as-limit, derivative-as-function, and derivative-as-local linear description all matter.

Why this belongs in a calculus book: many students can perform a derivative rule without being able to explain what the result means graphically or physically. This research supports a book design in which one derivative idea is revisited through tables, graphs, formulas, words, and applications instead of being taught once and abandoned.

Classroom link: when you solve a derivative problem, ask not only for the formula but also for the sign, the units, the graph behavior, and the local interpretation.

Excerpt 2. Foundational derivative ideas matter more than isolated rule fluency

Source: Manuel Santos-Trigo, Matías Camacho-Machín, and Fernando Barrera-Mora, *Focusing on foundational Calculus ideas to understand the derivative concept via problem-solving tasks that involve the use of a Dynamic Geometry System*, ZDM, 2024.
Link: https://link.springer.com/article/10.1007/s11858-024-01607-6

This work emphasizes that average rate of change, local linearity, and slope should be treated as a connected foundation for derivative learning. In other words, the derivative is not best learned as a sudden leap from formulas to rules. It is better learned as a tightening of ideas students have already met.

Why this belongs in a calculus book: the strongest derivative intuition often comes before the formal rule sheet. A student who can compare nearby changes, interpret slope, and recognize a local straight-line approximation is much better positioned to understand what the derivative is doing.

Classroom link: before differentiating mechanically, compare two nearby function values and explain what their quotient is trying to approximate.

Excerpt 3. The definite integral has many meanings, and students need the links between them

Source: Linda Poleth Montiel Buriticá, Gustavo Martínez Sierra, and Crisólogo Dolores Flores, *Exploring Conceptualisations of the Definite Integral in Mathematics Education: A Systematic Literature Analysis*, Educational Process, 2025.
Link: https://files.eric.ed.gov/fulltext/EJ1488983.pdf

This 2025 review examined literature from 2010 to 2024 and reported eight major conceptualizations of the definite integral: area under the curve, antiderivative, accumulation, accumulation function, average value, numerical approximation, real applications, and solids of revolution.

Why this belongs in a calculus book: if students meet the integral only as an antiderivative shortcut, they miss most of what makes the idea powerful. The review argues that instruction is often fragmented and that better teaching requires making the links explicit.

Classroom link: whenever you see a definite integral, ask which meaning is active in the problem. Is it net change, geometric area, average value, a numerical approximation, or a model of some accumulated quantity?

Excerpt 4. Active learning in calculus can improve autonomous work

Source: Javier Bilbao, Eugenio Bravo, Olatz García, and Carolina Rebollar, *Recovering autonomous work after the pandemic: analysis in Calculus for incoming Students in Technical Education degrees*, Humanities and Social Sciences Communications, published

December 24, 2024.
Link: https://www.nature.com/articles/s41599-024-04263-z

This study followed more than five hundred students and reported positive effects on autonomous work when active methodologies were used in first-year calculus settings.

Why this belongs in a calculus book: students do not only need correct explanations. They need a structure that helps them do mathematical work independently. That is one reason this book uses opening questions, exercise ladders, study tactics, and interpretation prompts instead of only theorem-plus-example format.

Classroom link: one of the best uses of homework is not repetition alone. It is staged independence: warm-up, core skill, interpretation, challenge, then modeling.

Excerpt 5. The Fundamental Theorem of Calculus becomes clearer when framed through quantities

Source: Suzanne White Brahmia and Patrick W. Thompson, *Framing the Fundamental Theorem of Calculus Through Physics-Based Quantities*, arXiv, January 8, 2025.
Link: https://arxiv.org/abs/2501.04219

This paper argues for explaining the Fundamental Theorem through linked ideas of change, rate, and accumulation instead of treating it as a symbolic bridge that suddenly appears after antiderivative rules.

Why this belongs in a calculus book: the theorem is easier to understand when the quantity being accumulated is visible. For example, if velocity is accumulated over time to produce displacement, then differentiating the accumulated total should recover the original velocity.

Classroom link: before writing $F(b) - F(a)$, name the quantity being accumulated and the units of the small product being added.

Excerpt 6. Calculus helps build personalized antibiotic dosing models

Source: Xiangqing Song, *Modeling of pharmacokinetic/pharmacodynamic parameters in regular intermittent intravenous infusion and translational application of the models in personalized antibiotics dosing*, Journal of Translational Medicine, published July 19, 2025.
Link: https://translational-medicine.biomedcentral.com/articles/10.1186/s12967-025-06832-5

This paper develops quantitative models for dosing effectiveness in intermittent intravenous infusion. Integrals and rate models are used to estimate exposure, target attainment, and dosage optimization in a personalized setting.

Why this is interesting: calculus here is not an academic exercise. It directly supports medical decisions about how much drug should be delivered and how long concentrations stay in a therapeutically useful range.

Classroom link: the moment an accumulation problem uses words such as concentration, exposure, or time above threshold, integral thinking is already in the room.

Excerpt 7. Machine learning is automating parts of pharmacokinetic model building

Source: Sam Richardson, Itziar Irurzun Arana, Andrzej Nowojewski, Diansong Zhou, Jacob Leander, Weifeng Tang, and collaborators, *A machine learning approach to population pharmacokinetic modelling automation*, Communications Medicine, published July 31, 2025.
Link: https://www.nature.com/articles/s43856-025-01054-8

This study reports an automated approach to population pharmacokinetic model development that can identify plausible model structures for multiple drug datasets in under two days on a moderate compute setup.

Why this is interesting: the underlying science still depends on calculus-based compartment models, differential equations, and optimization. Machine learning does not replace the calculus. It changes how rapidly large model spaces can be searched.

Classroom link: modern scientific computing often combines two questions: what differential model expresses the science, and what algorithm can fit or compare many candidate models efficiently?

Excerpt 8. PDE learning is moving toward coordinate-free representations

Source: Trung V. Phan, George A. Kevrekidis, Soledad Villar, Yannis G. Kevrekidis, and Juan M. Bello-Rivas, *Towards Coordinate- and Dimension-Agnostic Machine Learning for Partial Differential Equations*, arXiv, May 22, 2025.
Link: https://arxiv.org/abs/2505.16549

This paper explores learning PDE dynamics in a coordinate-free framework using exterior-calculus ideas. The goal is to make learned evolution laws transfer more naturally across dimensions, coordinate systems, and geometric settings.

Why this is interesting: multivariable calculus and vector calculus are often taught as if they only live on homework sets. Here they support a modern problem in scientific machine learning: how to represent physical laws without tying them to one arbitrary coordinate description.

Classroom link: changing coordinates should not change the underlying physical law. That idea is a deep reason vector and multivariable calculus are worth studying carefully.

Excerpt 9. Scientific machine learning is incorporating differential-algebraic systems

Source: Laurens R. Lueg, Victor Alves, Daniel Schicksnus, John R. Kitchin, Carl D. Laird, and Lorenz T. Biegler, *A Simultaneous Approach for Training Neural Differential-Algebraic Systems of Equations*, arXiv, April 7, 2025.
Link: https://arxiv.org/abs/2504.04665

This work studies neural differential-algebraic systems, where some parts of the model are learned from data while algebraic constraints and differential structure are retained explicitly.

Why this is interesting: many real models are not free-floating differential equations. They also contain constraints, balances, or conservation conditions. Calculus in practice often lives inside a larger system of conditions that must all be satisfied together.

Classroom link: optimization, differential equations, and constraints are not separate silos. In modern modeling they are often solved together.

Excerpt 10. Climate forecasting now uses kilometer-scale global models

Source: Ja-Yeon Moon, Jan Streffing, Sun-Seon Lee, Tido Semmler, Miguel Andrés-Martínez, Jiao Chen, and collaborators, *Earth's future climate and its variability simulated at 9 km global resolution*, Earth System Dynamics, published July 17, 2025.
Link: https://esd.copernicus.org/articles/16/1103/2025/esd-16-1103-2025.html

This paper presents future-climate simulations at roughly `9 km` global atmospheric resolution, allowing more detailed regional information about temperature, rainfall, winds, extreme events, and variability.

Why this is interesting: climate science is one of the clearest demonstrations that differential equations, numerical methods, approximation, and computation belong together. Planet-scale models are built from local laws of motion, transport, radiation, and thermodynamics, then solved approximately at enormous scale.

Classroom link: the numerical methods chapters of a calculus sequence are not side topics. They are part of how calculus reaches the real world.

Excerpt 11. Chemotherapy control can be framed as a delay-aware optimization problem

Source: *Delay-aware chemotherapy dosing via online critic learning*, Scientific Reports, 2025.
Link: https://www.nature.com/articles/s41598-025-25060-x

This paper develops an adaptive control method for chemotherapy dosing in a setting where treatment effects are delayed and patient conditions differ. The model combines differential-equation descriptions of tumor, immune, and healthy-cell dynamics with optimization logic that updates dosing decisions over time.

Why this is interesting: students often meet differential equations as if the final goal were only to solve for $y(t)$. In practice, the equation is often only the first layer. The real task is to use the model to choose when and how strongly to intervene.

Classroom link: every time a model includes a rate law plus a decision variable, calculus is already part of a control problem, not just a prediction problem.

Excerpt 12. Medication timing can be optimized against biological rhythms

Source: *Optimising anti-seizure medication timing using a dynamic network model of seizure rhythms*, Frontiers in Network Physiology, 2025.
Link: https://www.frontiersin.org/journals/network-physiology/articles/10.3389/fnetp.2025.1728848/full

This work incorporates absorption, elimination, and network-dynamics ideas into a model for seizure risk across multiple timescales. The key question is not only what dose to give, but when to give it relative to repeating rhythms in excitability.

Why this is interesting: calculus here supports timing, not only amount. Rates, delays, and periodic behavior combine in a setting where the medically relevant variable is dynamic risk over time.

Classroom link: when sinusoidal behavior, decay, and dosing schedules all appear in one problem, Chapters 10, 11, and 17 are suddenly talking to one another.

Excerpt 13. Quantum-computing research is still wrestling with nonlinear differential equations

Source: *Further improving quantum algorithms for nonlinear differential equations via higher-order methods and rescaling*, npj Quantum Information, 2025.
Link: https://www.nature.com/articles/s41534-025-01084-z

This paper studies algorithmic strategies for nonlinear differential equations in a quantum-computing setting. The point is not that calculus disappears in a more advanced computational framework. The point is that differential equations remain central enough that even new hardware paradigms are forced to confront them directly.

Why this is interesting: it shows the durability of the calculus core. Whether the computation is classical, statistical, or quantum, the need to model change by differential structure survives.

Classroom link: differential equations are not a detour out of mainstream mathematics. They are one of the main interfaces between calculus and computation.

Excerpt 14. Orbit design still depends on calculus-driven optimization

Source: *An analytical approach to generating orbital maneuver sets for spacecraft on-orbit services*, Scientific Reports, 2025.
Link: https://www.nature.com/articles/s41598-025-24686-1

This paper analyzes orbital maneuver windows and fuel-efficient transfers using mathematical relationships between terminal states, timing, and required velocity changes. The application is space operations, but the underlying logic is familiar: model the dynamics, identify a cost, and optimize under geometric and physical constraints.

Why this is interesting: students sometimes assume optimization chapters are mostly about boxes and fences because those are the first clean examples. In modern applications the same local-to-global logic governs astrodynamics, where timing and geometry are far more subtle.

Classroom link: a tangent-based local approximation and a constrained optimization problem can eventually turn into a spacecraft-design calculation.

Excerpt 15. Climate tipping models are built from coupled differential equations

Source: *High probability of triggering climate tipping points under current policies*, Earth System Dynamics, 2025.
Link: https://esd.copernicus.org/articles/16/565/2025/esd-16-565-2025.pdf

This study uses a compact system-dynamics model built from coupled differential equations to explore tipping risks under different warming scenarios. Even in a simplified setting, the interaction among feedback loops matters as much as any one equation by itself.

Why this is interesting: the model is a reminder that calculus often enters public policy through reduced systems that preserve the decisive feedback structure without pretending to resolve every physical detail.

Classroom link: equilibrium analysis, stability language, and coupled rates belong not only to homework problems but also to climate-risk reasoning.

Closing thought

One of the most important lessons of calculus is that local change and global structure are linked. These excerpts show the same lesson reappearing in different settings: teaching, medicine, machine learning, and climate science.

Appendix E. Chapter Projects and Modeling Labs

This appendix is designed for courses that want more than short end-of-section exercises. Each project is large enough for a written assignment, a recitation discussion, or a classroom presentation. The projects are grouped roughly in course order, but many can be adapted to different stages of a calculus sequence.

Using the projects well

A project should not feel like a longer homework problem with extra algebra. Good projects require students to make choices:

- which variables matter,
- what assumptions are reasonable,
- which calculus tools are appropriate,
- and how the answer should be interpreted in context.

For that reason, each project below is organized around a modeling question rather than a target formula. Instructors can shorten or expand the data requirements as needed.

Project 1. Estimating change from local data

Collect a small table of measurements from a process that changes over time. Good options include room temperature during the day, heart rate during exercise, or distance traveled during a walk. Use average rates of change on several intervals, then argue where the local rate appears largest and smallest.

Suggested deliverables:

- a table of data,
- a graph,
- two or three average-rate calculations,
- a short paragraph explaining where a tangent-line interpretation seems plausible,
- and one paragraph explaining the limits of the data.

This project works well early in Calculus I because it trains students to connect numerical evidence with the language of change before formal derivative rules dominate the course.

Project 2. Local linearity in mental estimation

Choose five quantities people estimate mentally in real life, such as:

- $\sqrt{15.8}$,
- $1/49$,
- $e^0.08$,
- $\sin(0.12)$,
- or $(1.02)^5$.

For each quantity, choose a nearby easy base point, build a linear approximation, compare with the exact value, and report the error. The report should explain why the chosen base point is good, not just whether the answer is accurate.

A strong extension is to compare two base points and decide which one produces the better approximation. This project makes the phrase "local linearity" operational rather than rhetorical.

Project 3. Product-rule models in economics or biology

Find a quantity that is naturally expressed as a product of two changing factors. Examples include:

- revenue equals price times quantity sold,
- biomass equals average mass times population,
- kinetic energy equals $(1/2)mv^2$,
- or signal power equals voltage times current.

Write the quantity as a product, differentiate it, and interpret each term in the product rule. The goal is to explain what the two terms mean separately, not merely to compute the derivative.

This project is effective because it shows that derivative rules encode interpretable mechanisms. One term records change in the first factor while the second stays fixed; the other term records the reverse.

Project 4. Related rates from geometric constraints

Choose one moving geometry scenario:

- a ladder sliding down a wall,
- a spherical balloon being inflated,
- a cylindrical tank filling with water,
- or a camera drone moving away from a target.

Identify all variables, write the geometric constraint, differentiate with respect to time, and solve for the requested rate. The report must include a labeled diagram and a paragraph explaining why one variable cannot change independently of the others.

Students often discover here that drawing the geometry clearly is not optional. The diagram is part of the mathematics.

Project 5. Optimization under a design constraint

Create a design problem with one quantity to maximize or minimize. Example themes:

- enclosing the largest area with a fixed fence length,
- minimizing material cost for a box,
- maximizing lighting coverage from a wall-mounted fixture,
- or finding the least-cost packaging design under a volume requirement.

The project should include:

- a description of the real-world design constraint,
- a function expressing the quantity to be optimized,
- derivative analysis,
- a justification that the critical point actually solves the problem,
- and a final design recommendation in plain language.

This project works best when students are required to explain how the calculus result would matter to a real decision-maker.

Project 6. Net change from a measured rate

Start with a rate function or rate table for a realistic process:

- fuel flow into a tank,
- electrical power output,
- drug infusion rate,
- rainfall intensity,
- or website traffic per minute.

Interpret the definite integral of the rate over a chosen interval, compute or estimate it, and compare net change with total variation if the rate changes sign.

The key learning objective is to prevent students from treating the integral as a decorative symbol. They must say what the units are and what the accumulated quantity means.

Project 7. The Fundamental Theorem in motion

Build a short presentation around one statement:

> The derivative of an accumulation function is the current rate of accumulation.

Support that statement with:

- one geometric example,
- one motion example,
- one units-based explanation,
- and one symbolic example involving a variable upper limit.

This project is especially useful in recitation or peer instruction because students tend to remember the theorem better when they can explain it in several languages: graphical, numerical, contextual, and symbolic.

Project 8. Integration method comparison

Choose ten integrals from Chapters 7 and 8. For each one:

- state the best method,
- justify the choice,
- identify one tempting but inferior method,
- and explain how you knew the chosen method had succeeded.

The final product should look like a strategy manual rather than a solution key. The purpose is to build method-recognition fluency, which is one of the main reasons advanced integration chapters in traditional textbooks are so long.

Project 9. Numerical integration with experimental data

Collect or simulate a small data set for a quantity that varies over time or distance. Apply the trapezoidal rule and Simpson's Rule where appropriate. Then answer:

- Which estimate seems more reliable?
- How does graph shape influence the error?
- What would additional data points change?

This project works particularly well in a lab setting because students see that calculus is useful even when no closed-form formula is available.

Project 10. Slicing a volume two ways

Choose a solid whose volume can be computed by two different methods, such as washers and shells, or vertical and horizontal slicing. Set up both integrals, compute both, and explain why they agree.

This project deepens geometric understanding because it forces students to see that different symbolic expressions can represent the same physical object.

Project 11. Mass, moment, and center of mass

Construct a density model for a rod, lamina, or wire. Compute:

- total mass,
- one or more moments,
- and the center of mass.

Then write a short interpretation of the result. Does the center of mass lie where visual intuition suggests? If not, why not?

This project is especially useful because it turns a new formula into a direct extension of the accumulation idea from earlier chapters.

Project 12. Series as approximation engines

Choose one function that is expensive or inconvenient to compute exactly, such as:

- $\sin x$,

- e^x,
- $\ln(1 + x)$,
- or `1/(1 − x)`.

Build several Taylor polynomial approximations, compare them at selected inputs, and explain how the interval of reliability changes with degree. A strong version of the project includes error discussion and a graph comparing the function with its approximations.

This project makes explicit why series chapters belong in a calculus sequence: they provide a systematic way to replace hard functions by easier polynomial ones.

Project 13. Differential-equation model critique

Take a standard first-order model such as exponential growth, logistic growth, or Newton's law of cooling. Fit the model to a plausible scenario, solve it, and then critique the assumptions.

Questions to include:

- Which assumptions are realistic?
- Which are clearly simplifications?
- What changes in the real system would break the model?

The project teaches a vital habit for later mathematics and science: a solved differential equation is not the same thing as a trustworthy model.

Project 14. Multivariable approximation in context

Choose a function of two variables from economics, physics, or environmental science. Compute partial derivatives, the gradient, and a linear approximation at a selected point. Then interpret:

- which variable has the stronger local effect,
- what the gradient direction means,
- and how the linear approximation could be used in a quick estimate.

This appendix includes the project because multivariable calculus often becomes abstract quickly; contextual approximation helps ground the ideas.

Project 15. Circulation and flux field study

Construct or simulate a vector field that represents wind, fluid flow, traffic, or another directional process. Then describe:

- what circulation would measure,
- what flux would measure,
- and how line or surface integrals capture those quantities.

Even if the full computations are modest, the interpretation work prepares students for the conceptual leap into vector calculus.

Assessment notes for instructors

Projects are often most useful when graded on several dimensions rather than one:

- mathematical setup,
- correctness of calculus,
- quality of explanation,
- unit interpretation,
- and reflection on assumptions.

Students should be rewarded for modeling judgment, not only for perfect algebra. In a classroom text, that distinction matters because calculus is not merely a computation course; it is also a course in representing change and accumulation carefully.

Appendix F. Proof and Writing Workshop

Many students can compute correctly long before they can explain clearly. Yet advanced calculus, upper-division mathematics, and technical writing all demand more than symbolic success. This appendix is a practical guide to writing solutions, explanations, and short proofs that sound mathematically mature.

Why mathematical writing matters

A proof is not a stream of equations. A good solution does at least three things:

- states what is being shown,
- explains why each step is valid,
- and signals how the steps fit the larger argument.

In calculus, weak writing often hides weak thinking. Students may have the right derivative formula or integral evaluation, but their work still leaves important questions unanswered:

- What quantity is being computed?
- Why was this method appropriate?
- What assumptions were used?
- What does the result mean?

Strong writing forces those questions into the open.

A basic template for complete solutions

For most classroom problems, a complete solution can be organized in four parts:

1. Restate the task in mathematical language.
2. Choose the governing idea or formula.
3. Carry out the calculation with enough explanation to show why it is legitimate.
4. Interpret the result, including units or geometric meaning when relevant.

Here is a short example.

Problem: Find the area between $y = x$ and $y = x^2$ on `[0, 1]`.

Weak solution:

$\int_0^1 (x - x^2)dx = 1/6.$

Better solution:

On `[0, 1]`, the line $y = x$ lies above the parabola $y = x^2$, so the area is

$A = \int_0^1 (x - x^2)dx.$

Evaluating,

$A = [x^2/2 - x^3/3]_0^1 = 1/2 - 1/3 = 1/6.$

Therefore the enclosed area is $1/6$ square units.

The second version is only slightly longer, but it tells the reader how the integral was chosen and what the answer represents.

Explaining derivative work

Derivative solutions often become unreadable because students skip the structural diagnosis. When a derivative uses the product rule, quotient rule, or chain rule, say so explicitly.

Example:

Find the derivative of

$f(x) = (x^2 + 1)^4.$

A strong explanation says:

The function is a composition, with outer function u^4 and inner function $u = x^2 + 1$. By the chain rule,

$f'(x) = 4(x^2 + 1)^3(2x) = 8x(x^2 + 1)^3.$

This style of explanation signals that the answer came from a recognized structure, not from accidental algebra.

Explaining integrals

Integral writing benefits from the same discipline. Do not begin with computation if the central difficulty is method choice or model setup.

For instance, if the task is to compute

$\text{int} \times e^x dx,$

a well-written solution says:

This is a product of an algebraic function and an exponential function, so integration by parts is natural. Let $u = x$ and $dv = e^x dx$. Then $du = dx$ and $v = e^x$, so

$\text{int} \times e^x dx = xe^x - \text{int} e^x dx = e^x(x - 1) + C.$

The sentence before the algebra is not decorative. It shows that the writer understands why the chosen method fits.

Writing about limits

Limit arguments require precision in language. Phrases like "gets close" are fine for intuition, but formal explanations need clearer statements.

Instead of writing:

$f(x)$ gets close to `3`,

write:

$f(x)$ approaches `3` as `x` approaches `2`, meaning that by taking `x` sufficiently near `2`, the values of $f(x)$ can be made arbitrarily close to `3`.

The exact level of rigor depends on the course, but the habit of replacing vague closeness language with conditional statements is valuable early.

Proof skeletons that recur in calculus

Many proof tasks in a calculus sequence follow one of a small number of patterns.

Pattern 1: prove a property from a definition

Examples:

- continuity from a limit definition,
- differentiability implying continuity,
- or convergence of a sequence from the epsilon definition.

The writing strategy is:

1. state the definition,
2. name the given information,

3. show directly how the definition is satisfied.

Pattern 2: prove a formula by transforming a known identity

Examples:

- integration by parts from the product rule,
- quotient rule from the product rule plus reciprocal differentiation,
- or trigonometric identities used in integration.

The writing strategy is:

1. begin from the known statement,
2. manipulate it lawfully,
3. end at the target statement,
4. explain why each transformation is legitimate.

Pattern 3: prove uniqueness or existence under an added condition

Examples:

- initial value problems selecting one antiderivative,
- optimization problems producing a unique maximizing input,
- or differential-equation models with specified initial data.

The writing strategy is:

1. construct the general family,
2. impose the condition,
3. show that exactly one member satisfies it.

Recognizing these patterns helps students feel that proofs are learnable structures rather than mysterious acts of inspiration.

Sentence starters for mathematical explanation

Students often know what they want to say but not how to begin. The following sentence stems are useful and professional without sounding artificial:

- "Because the function is continuous on the interval, ..."
- "The expression has the form of a composition, so ..."
- "To model the total quantity, consider a representative slice ..."

- "Differentiating both sides with respect to time gives ..."
- "Since the integrand is positive on the interval, ..."
- "The units indicate that this integral represents ..."
- "This result is reasonable because ..."

These stems help move a solution from raw symbolism to readable argument.

Common writing problems and their repairs

Problem: equations with no subject

Weak:

$$= 4x^3 - 2$$

Better:

$$f'(x) = 4x^3 - 2$$

Every displayed equation should belong to a sentence or state a clear mathematical claim.

Problem: unexplained variable switches

Weak:

$$\int_0^1 2xe^{x^2}dx = \int_0^1 e^u du$$

Better:

Let $u = x^2$, so $du = 2x\,dx$. Because $u = 0$ when $x = 0$ and $u = 1$ when $x = 1$,

$\int_0^1 2xe^{x^2}dx = \int_0^1 e^u du.$

The better version tells the reader where `u` came from and why the bounds did not remain in `x`.

Problem: formulas with no interpretation

Weak:

$W = 48.$

Better:

$W = 48$ joules, so the variable force performs `48` units of work over the stated interval.

Interpretation is not extra. It is part of what makes the answer complete.

Writing proofs of "why this formula is reasonable"

In a first calculus course, many proofs are not fully formal, but they can still be good mathematics. A useful mode is the "reasonableness proof," where the goal is to explain convincingly why a formula should have the form it does.

Example: arc length.

One can write:

On a short interval, the curve is approximately a line segment. For a small change Δx, the vertical change is approximately $f'(x)\Delta x$, so the segment length is approximately

$$\mathrm{sqrt}((\Delta x)^2 + (f'(x)\Delta x)^2) = \mathrm{sqrt}(1 + (f'(x))^2)\Delta x.$$

Adding these short lengths and passing to the limit yields the arc-length formula.

This is not a formal proof in the style of a real analysis course, but it is a mathematically serious explanation because it identifies the small model, the approximation, and the limiting process.

When to use words and when to use symbols

Strong mathematical writing alternates between words and symbols rather than choosing one exclusively.

Use words for:

- naming the method,
- describing the geometry,
- explaining units,
- and stating the conclusion.

Use symbols for:

- the calculation itself,
- the exact formula,
- and compact logical relationships.

A good page of calculus typically moves back and forth between the two.

A checklist for final proofreading

Before turning in a proof or solution set, check the following:

- Are all variables defined?
- Does each displayed equation belong to a sentence?
- Have all substitutions and method choices been explained?
- Are units included when the context requires them?
- Does the final sentence state what the answer means?
- Have you distinguished between approximate and exact statements?
- If a theorem is used, have you named its hypotheses?

This checklist is intentionally simple. Most weak mathematical writing improves sharply when these items are checked carefully.

Practice prompts for writing

The following prompts are useful for recitation or as revision exercises:

1. Explain in words why the chain rule is necessary for nested functions.
2. Write a paragraph interpreting the definite integral of a rate function with units.
3. Compare a weak and strong explanation of substitution in an integral.
4. Write a short proof that different antiderivatives of the same function differ by a constant.
5. Explain why a center-of-mass formula is a weighted average.
6. Describe the difference between an indefinite integral and a definite integral without using the phrase "one has bounds and one does not."

These prompts matter because mathematical maturity grows through explanation, not only through repetition of computations.

Final note

Good calculus writing is not ornament placed on top of finished mathematics. It is one of the ways mathematical understanding becomes visible. A student who can explain why a method applies, why the units fit, and why the answer is reasonable is usually a student who understands the subject at a much deeper level.

Appendix Q. Concept Checks and Writing Prompts

This appendix adds a chapter-organized bank of short conceptual questions and writing prompts. Major commercial calculus texts often include these throughout sections and at chapter ends; this appendix collects them in one assignable place until the main-chapter embedding is expanded further.

Chapter 1. Functions and change

1. Explain why a rate needs units while a raw quantity may not communicate change at all.
2. Write one paragraph distinguishing a function, a formula, and a graph.
3. Describe a real setting in which average rate of change is useful but hides important local behavior.

Chapter 2. Limits and continuity

1. Explain why a limit can exist even when the function is undefined at the point.
2. In words, compare a removable discontinuity and a jump discontinuity.
3. Write a short explanation of why matching values along a few paths does not prove a multivariable limit exists.

Chapter 3. Derivatives

1. Explain why a derivative is a function and not merely a number.
2. Describe the geometric meaning of a negative derivative at a point.
3. Give a physical interpretation of the second derivative in a motion setting.

Chapter 4. Derivative rules

1. Explain the chain rule without symbols by using the language of nested dependence.
2. Describe why implicit differentiation is not merely an algebra trick.
3. Write a paragraph explaining when linearization is useful and when it should not be trusted.

Chapter 5. Derivative-based behavior

1. Explain the difference between a critical point and an extremum.
2. Describe how concavity changes the visual meaning of an increasing function.
3. Write a short memo explaining why optimization problems often require a final context check after differentiation.

Chapter 6. Definite integrals

1. Explain the difference between total area and signed area.
2. Describe how a Riemann sum encodes the idea of local contributions accumulating into a global total.
3. Write a short explanation of average value over an interval and how it differs from an arithmetic mean of sampled values.

Chapter 7. The Fundamental Theorem

1. Explain why antiderivatives come in families rather than as unique formulas.
2. Describe how the Fundamental Theorem links rates and accumulation.
3. Write a paragraph explaining why substitution is a structural recognition method rather than a symbol-replacement trick.

Chapter 8. Integration techniques

1. Explain how you decide between substitution, integration by parts, and partial fractions.
2. Describe what numerical integration contributes when exact antiderivatives are unavailable.
3. Write a short reflection on why method choice is one of the biggest conceptual hurdles in Calc II.

Chapter 9. Applications of integration

1. Explain why a volume problem begins with choosing a local slice or shell before any integration happens.
2. Compare area, mass, and work as three versions of one accumulation story.
3. Write one paragraph describing how units can detect an incorrect integrand in an applications problem.

Chapter 10. Sequences and series

1. Explain why $a_n \to 0$ is necessary but not sufficient for $suma_n$ to converge.
2. Describe the difference between absolute and conditional convergence in words.
3. Write a short explanation of why Taylor polynomials belong in calculus rather than in a separate approximation course.

Chapter 11. Differential equations

1. Explain why a slope field can be valuable even before an exact solution is known.
2. Describe how an equilibrium can be stable, unstable, or neither.
3. Write a paragraph comparing model error and numerical error.

Chapter 12. Vectors and space

1. Explain why velocity and speed are not the same quantity.
2. Describe what projection measures geometrically.
3. Write a short explanation of why the cross product naturally encodes area and orientation.

Chapter 13. Multivariable functions

1. Explain why the gradient is perpendicular to level curves.
2. Describe the tangent plane as a multivariable linear model.
3. Write a short explanation of why saddle points matter in optimization.

Chapter 14. Multiple integration

1. Explain why changing the order of integration is a geometric decision.
2. Describe why polar coordinates require the factor `r`.
3. Write a short paragraph comparing mass integrals and probability-density integrals.

Chapter 15. Vector calculus

1. Explain the difference between circulation and flux.
2. Describe how Green's Theorem, the Divergence Theorem, and Stokes' Theorem resemble the Fundamental Theorem of Calculus.
3. Write a short explanation of why orientation errors are so common in vector calculus.

Chapter 16. Parametric curves, polar coordinates, and curvature

1. Explain why a parametrized curve contains more information than the eliminated rectangular equation alone.
2. Describe why polar coordinates are natural for origin-centered geometry.
3. Write a short explanation of what curvature measures that slope does not.

Chapter 17. Second-order differential equations and oscillation

1. Explain why a second-order model usually needs both initial position and initial velocity.
2. Describe how damping changes the behavior of an oscillating system.
3. Write a short paragraph comparing resonance with ordinary forced response.

Chapter 18. Improper integrals and long-tail behavior

1. Explain why an improper integral must be rewritten as a limit before it can be evaluated correctly.
2. Describe the difference between a function that tends to zero and a function whose improper integral converges.
3. Write a short paragraph explaining how improper integrals connect accumulation to probability or long-tail risk.

Instructor use notes

- Use the odd-numbered prompts as warm-up discussion or exit-ticket items.
- Use the even-numbered prompts as paragraph-writing homework or oral-exam preparation.
- Pair a concept-check prompt with a computational set whenever a section risks becoming purely procedural.

Appendix R. Technology Explorations and Computing Labs

This appendix adds technology-centered explorations that are common in large college calculus texts. The goal is not to require a specific platform, but to make room for graphing calculators, spreadsheets, symbolic systems, and lightweight numerical tools.

Each lab can be completed with graphing software, a spreadsheet, or a computer algebra system unless otherwise noted.

Lab 1. Average versus local change from data

Collect or simulate a data table for a changing quantity such as water level, stock price, or position. Compute average rates of change on several intervals, then estimate local behavior with shorter intervals. Write a short note on when the average-rate story becomes misleading.

Lab 2. Visualizing limits numerically

Create a table and graph for a function with a hole, such as $(x^2 - 4)/(x - 2)$. Use values approaching the target from both sides and compare the numerical evidence with the algebraic simplification.

Lab 3. Tangent-line approximation

Graph a function together with its tangent line at a chosen point. Use a slider or table to compare the exact function and the tangent-line estimate as the distance from the base point changes. Record where the linear approximation is convincing and where it fails.

Lab 4. Chain-rule dependency map

Choose a multistage model from biology, economics, or geometry and build a dependency diagram showing how one variable changes another. Then compute a related chain-rule derivative and explain what each factor means.

Lab 5. Optimization with graph support

Use graphing technology to study an optimization problem before taking derivatives. Plot the objective function, identify plausible maxima or minima, and then confirm analytically. Compare the graphical guess with the exact result.

Lab 6. Riemann-sum convergence

Use a spreadsheet or CAS to compute left, right, and midpoint Riemann sums for the same function on the same interval. Increase the number of subintervals and observe how the sums behave. Write a short summary comparing convergence speed.

Lab 7. Accumulation function experiment

Define an accumulation function numerically from sampled rate data. Plot both the original rate and the resulting accumulation. Explain how changes in the rate graph control the shape of the accumulation graph.

Lab 8. Integration-method comparison

Choose one definite integral and approximate it with left sums, midpoint sums, trapezoids, and Simpson-style software output if available. Compare the answers and report which method appears most efficient for the chosen integrand.

Lab 9. Solids of revolution visualization

Use graphing software to visualize a region and the solid formed when it is revolved about an axis. Compare shell and washer descriptions for the same solid and write a short explanation of why one setup is cleaner.

Lab 10. Partial sums and convergence dashboards

For three series with different behaviors, compute and plot partial sums. Include one convergent geometric series, one divergent harmonic-type series, and one alternating series.

Explain what a student can learn from the graph that is harder to see from the formula alone.

Lab 11. Taylor approximation comparison

Plot a function together with several Taylor polynomials centered at the same point. Compare how the approximation interval changes with degree. Include a short section on whether a high-degree polynomial is always the best practical choice.

Lab 12. Slope fields and numerical solutions

Generate a slope field for a first-order differential equation and overlay numerical or exact solution curves if available. Use this to explain how local derivative information predicts global trends.

Lab 13. Euler-method step size study

Run Euler's Method on the same initial value problem with at least three different step sizes. Compare the resulting tables and graphs. Write a short paragraph separating truncation error from model error.

Lab 14. Contour maps and gradients

Use a graphing system to display a surface and its contour map. Estimate gradient directions from the contours, then confirm by computing partial derivatives and plotting the gradient at selected points.

Lab 15. Double-integral region design

Choose a nonrectangular planar region and represent it in both rectangular and polar coordinates if possible. Use technology to sketch the region, then explain which coordinate system better matches the geometry and why.

Lab 16. Vector-field atlas

Build a small gallery of vector fields that includes a radial field, a rotational field, and a conservative field. For each one, state whether circulation or flux is the more natural quantity to study and justify the answer visually.

Lab 17. Surface normals and flux orientation

Visualize a simple surface patch together with two opposite normal directions. Use the picture to explain why reversing orientation changes the sign of a flux integral without changing the shape of the surface.

Lab report template

For any lab in this appendix, the report should include:

1. the question being explored,
2. the equations or data used,
3. the visual output or table,
4. the mathematical interpretation,
5. and one sentence on what the technology clarified that pencil-and-paper work alone did not.

Instructor use notes

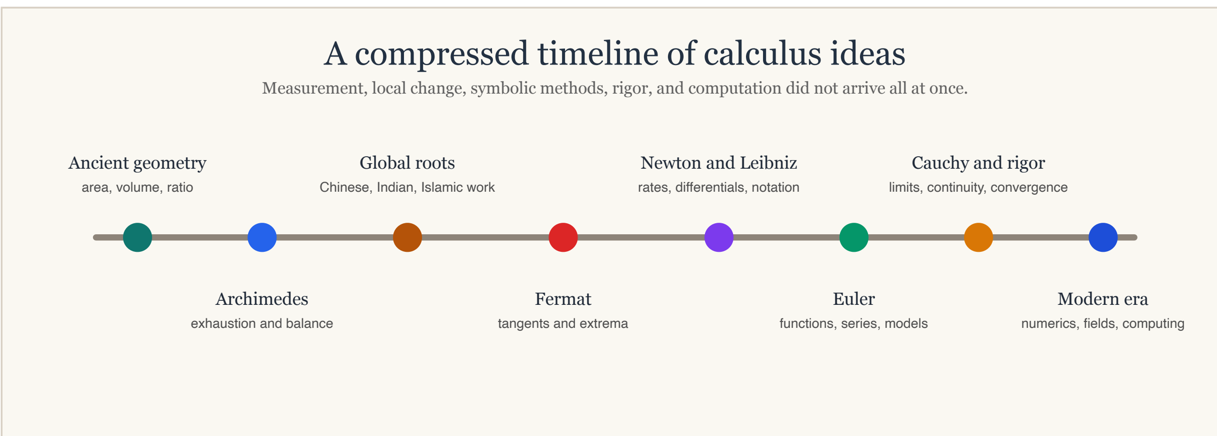
- Assign one lab every one or two chapters in a technology-supported course.
- Use Labs 6, 10, 11, and 13 as recurring numerical-analysis threads.
- Use Labs 14 through 17 to strengthen spatial intuition in Calc III.

Appendix Z. Historical Interludes and Milestones

This appendix is not a complete history of calculus. It is a short set of interludes designed to answer a different question:

Why do these ideas look the way they do?

Calculus did not appear all at once, and it did not appear from one source alone. Problems about area, volume, tangent lines, motion, approximation, and infinite processes were worked on across many centuries and in many mathematical traditions. The modern course compresses that long development into a sequence of chapters. History re-expands the story and makes the underlying choices easier to see.



1. Before calculus had a name

Long before the word *calculus* became standard, mathematicians were already studying the raw ingredients of the subject.

- Greek geometry developed rigorous arguments about length, area, volume, and proportion.
- Chinese mathematical texts developed sophisticated procedures for measurement, approximation, and solving practical quantitative problems.
- Indian mathematicians developed strong trigonometric methods and, in the Kerala school, remarkable series expansions related to later power-series ideas.
- Islamic mathematicians preserved, critiqued, extended, and transmitted major parts of the Greek mathematical tradition while also deepening algebraic and trigonometric methods that later fed directly into early modern European mathematics.

This matters because the modern course can accidentally suggest that calculus is just a list of formulas invented in one dramatic moment. It is better to see the subject as a convergence of several older habits:

- compare nearby values,
- slice complicated shapes into simpler pieces,
- accumulate many small contributions,
- search for patterns that remain stable under limiting refinement.

Those are the habits students still use now.

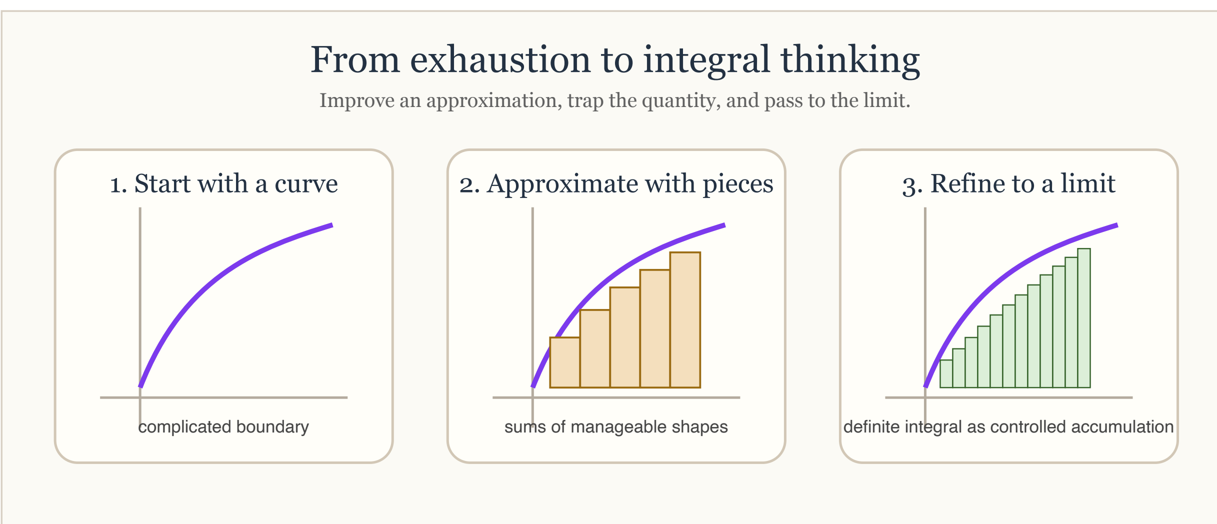
2. Archimedes and the exhaustion tradition

Archimedes stands near the beginning of the calculus story because he repeatedly attacked area and volume problems by replacing a curved object with many simpler pieces whose total could be controlled. The method is now described as a method of exhaustion.

The central idea is simple to state:

1. build an inscribed or circumscribed approximation,
2. improve it systematically,
3. trap the true quantity between values that squeeze together.

That pattern is already recognizably calculus. It is not yet a modern limit definition, but the logic is close. Archimedes used it to study areas bounded by curves and volumes of solids, and he treated geometry with unusual daring and precision.



Two classroom lessons come from this historical thread.

First, the definite integral is not merely an antiderivative machine. Historically and conceptually, it arises from trying to measure complicated shapes by controlled approximation.

Second, the foundations of area and volume matter. When this book treats area and volume as finitely additive functions with geometric normalization rules, it is echoing a very old tradition: before one can integrate well, one must be clear about what is being measured and why the measurement behaves predictably under decomposition.

Archimedes also reminds us that elegance in mathematics often begins as a measurement problem. The route from geometry to analysis is not artificial. It is one of the main roads by which calculus entered mathematics.

3. Tangents, extrema, and the road to derivatives

The derivative chapter can feel modern because students meet it through notation such as `f'(x)` and limit definitions. But the underlying problems are older than the notation.

Questions like these came first:

- How can one find a tangent to a curve?
- How can one locate a maximum or minimum value?
- How can one describe instantaneous speed rather than average speed?

Pierre de Fermat worked on methods that foreshadowed differential reasoning, especially in tangent and optimization problems. He did not write a modern derivative definition, but he did search for algebraic procedures that isolate what changes and what stays fixed when two nearby quantities are compared. In that sense he belongs to the prehistory of derivatives.

This is a useful corrective for students. Derivatives were not invented because someone wanted a new symbol. They were invented because existing geometry and motion problems demanded a cleaner language for local change.

That is why this book repeatedly frames the derivative as several linked ideas:

- slope,
- rate of change,
- local linear model,
- limiting ratio.

History shows that no single one of these ideas was enough by itself. The power of calculus came from bringing them together.

4. Newton and Leibniz: two entry points, one subject

Any brief history of calculus eventually reaches Isaac Newton and Gottfried Wilhelm Leibniz. They are often presented as rivals, but pedagogically it is more helpful to see them as emphasizing two complementary entry points.

Newton leaned into motion. His language of *fluents* and *fluxions* treated changing quantities dynamically. A variable was something that flowed; its derivative-like quantity expressed how quickly it flowed. This perspective still lives in modern physics, differential equations, and rate-based modeling.

Leibniz leaned into notation and structure. His differential notation dx, dy, and int proved extraordinarily fertile because it organized computation, made patterns visible, and encouraged a systematic symbolic language for change and accumulation.

Students still feel both traditions:

- when they interpret dy/ dx as a rate, they are close to Newton,
- when they use substitution or integration by parts in symbolic form, they are close to Leibniz.

The best calculus teaching does not force a choice between them. It keeps both viewpoints alive. One explains why the subject matters; the other explains why the algebra works so well.

5. Euler and the expansion of the subject

If Newton and Leibniz helped launch calculus, Leonhard Euler helped normalize its everyday language. He pushed forward notation, functions, differential equations, infinite series, and applications with unusual breadth. Many of the formulas that students now treat as standard objects were clarified or popularized in the eighteenth century through Euler's work and the work of those around him.

Euler matters for a classroom reason that goes beyond biography. He shows what happens when calculus becomes a general method rather than a collection of isolated tricks. In his hands, the same symbolic language moved across geometry, mechanics, series, and differential equations.

That is also why long calculus textbooks are long. A serious text does not stop after the first successful derivative and integral examples. It keeps revealing the same ideas in new settings:

- functions and graphs,

- optimization,
- differential equations,
- infinite series,
- multivariable models,
- geometry of fields and flows.

The subject became large historically because its methods kept transferring.

6. Cauchy, rigor, and the demand for hypotheses

Early calculus was powerful, but not always precise by modern standards. Arguments involving infinitesimals, convergent series, and continuity often worked, but not always with clearly stated conditions. During the nineteenth century, Augustin-Louis Cauchy and many others pushed the subject toward sharper definitions and better control over assumptions.

This matters directly for how a textbook states theorems. Phrases such as "under the usual conditions" can hide essential hypotheses. History explains why mathematicians became less tolerant of that vagueness. Without careful assumptions, one can get formally similar calculations that lead to false conclusions.

For students, rigor should not be confused with hostility or obscurity. The real function of rigor is trust. A well-stated theorem tells the reader:

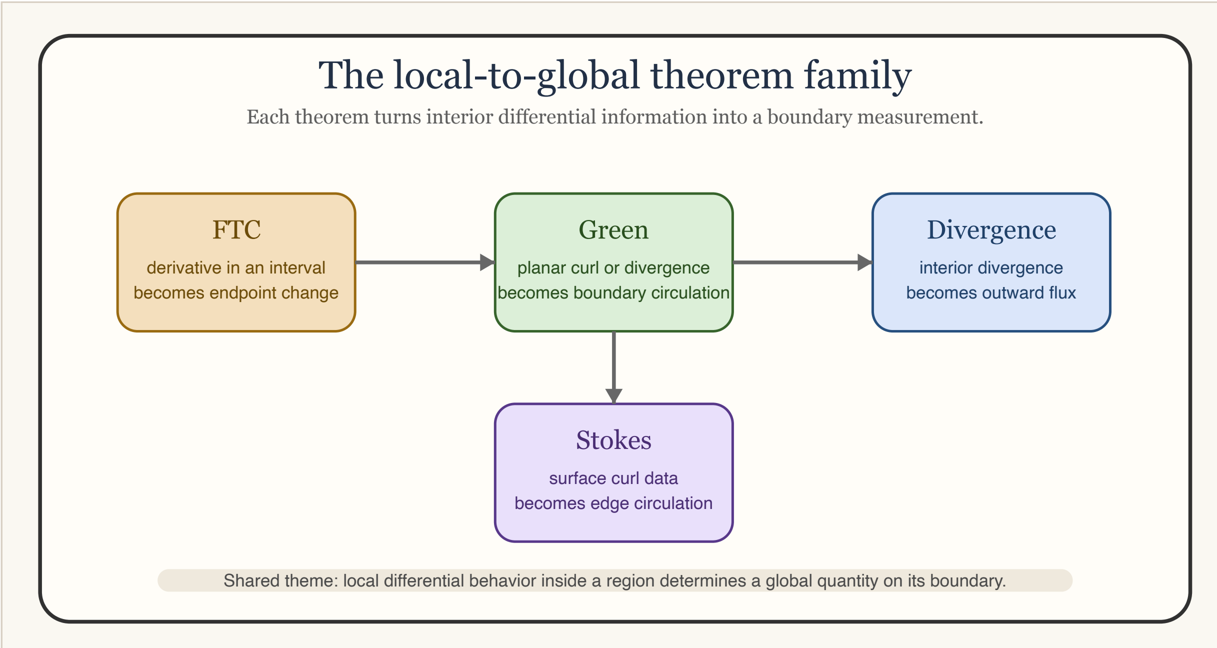
- what assumptions are required,
- what conclusion follows,
- where the method is safe to use.

That is why this book's later revisions tightened theorem statements in the Fundamental Theorem of Calculus, the multivariable second-derivative test, and the major vector-calculus theorems. Those edits were not cosmetic. They moved the manuscript closer to the historical lesson Cauchy helped teach: clarity about assumptions is part of mathematical understanding.

7. Area, circulation, flux, and the local-to-global family

The later chapters of a calculus book often feel far away from the opening material. Students can leave with the false impression that Green's Theorem, the Divergence Theorem, and Stokes' Theorem are advanced ornaments attached to an otherwise unrelated course.

Historically, they are better seen as members of a family. Each theorem tells us that local differential behavior accumulated across a region can be converted into a boundary measurement.



The family resemblance looks like this:

- the Fundamental Theorem connects derivative information to endpoint change,
- Green's Theorem connects planar curl/divergence information to circulation or flux around a boundary,
- the Divergence Theorem connects interior divergence to outward flux across a closed surface,
- Stokes' Theorem connects surface curl information to circulation around the edge.

This line of development carries the subject from one dimension to higher dimensions without abandoning its central philosophy. Local behavior still builds global structure. Only the geometry changes.

For classroom use, this historical perspective helps students avoid one of the biggest late-course mistakes: memorizing each theorem as an isolated template. It is better to see them as repeated instances of one big idea.

8. Numerical calculus and the computer era

Many students treat numerical methods as secondary because symbolic calculus is often introduced first. Historically, that ordering is misleading. Approximation has always been central.

Archimedes approximated geometry. Early modern astronomers and mechanicians needed numerical estimates. Differential equations quickly outran the range of exact formulas available in closed form. By the time modern science, engineering, and computing arrived, numerical calculus was not an optional supplement. It was indispensable.

That is why contemporary calculus texts spend so many pages on material that may look "extra" at first glance:

- numerical integration,
- Euler methods and improved Euler methods,
- error control,
- power-series approximations,
- coordinate changes chosen to simplify computation,
- technology-supported visualization.

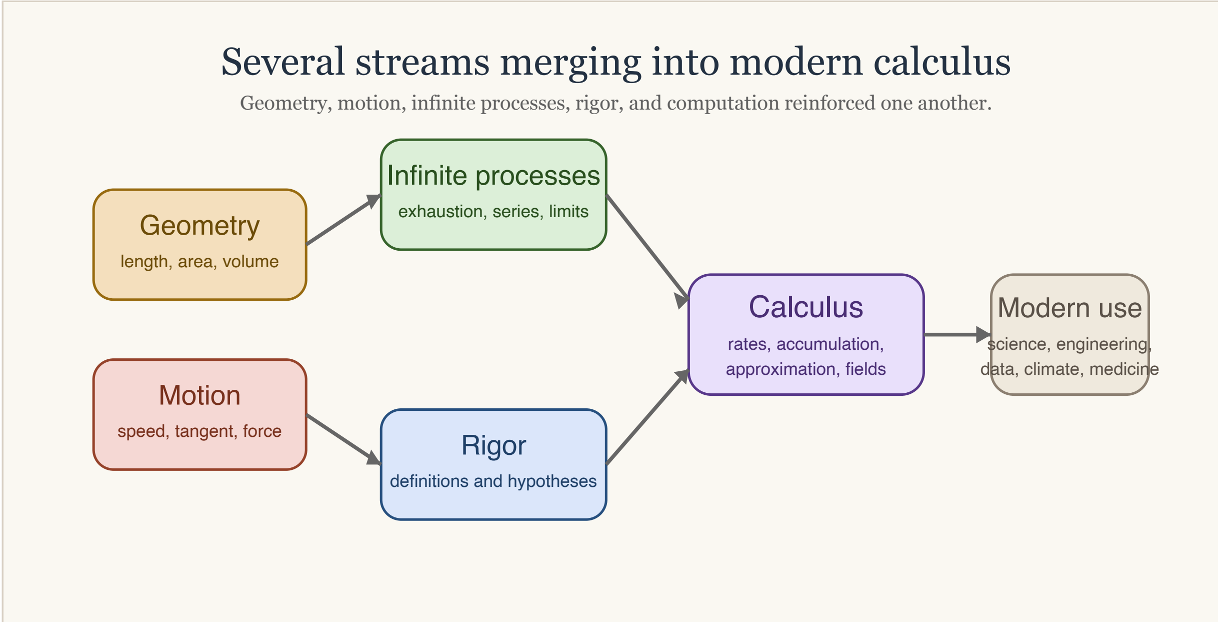
Large textbooks are not large only because they include more definitions. They are large because they support the student in moving among exact formulas, approximations, graphs, tables, and computational checks.

When this book adds more worked examples and more figures, it is following that historical reality. Calculus became powerful partly because it can be done in more than one mode.

Additional interlude. Berkeley's criticism and Maclaurin's response

In the eighteenth century, not everyone trusted the logical status of calculus. George Berkeley famously criticized the new methods for relying on quantities that seemed to be treated as both zero and not zero depending on what the calculation needed. His critique was rhetorically sharp because it targeted a real weakness: the methods were powerful, but their foundations were not yet stated in a way that later mathematicians would accept as fully satisfactory.

Colin Maclaurin's *Treatise of Fluxions* is historically important because it shows a serious attempt to answer that criticism by rebuilding calculus with more explicit geometric care. He appealed backward to ancient exhaustion methods while also participating in the forward motion of modern analysis.



This episode matters in the classroom for a practical reason. Students often experience theorem hypotheses and precise definitions as annoying extra rules added after the "real math" is done. The Berkeley-Maclaurin episode shows the opposite. Precision grows out of the need to defend and clarify methods that otherwise remain vulnerable to confusion.

Additional interlude. Fourier and the physical meaning of infinite series

One reason series chapters in calculus textbooks become long is that they do two jobs at once:

- they study convergence as a mathematical question,
- and they show how infinite expansions become tools for modeling heat, waves, vibration, and approximation.

Joseph Fourier sits near the center of that expansion. His work on heat flow made it impossible to treat trigonometric series as merely formal manipulations. Series now had to answer to physics. They had to approximate real temperature distributions and real time evolution.

For students, this is a useful historical corrective. Power series and trigonometric series are not in the book only because analysts enjoy expanding functions. They are in the book because local algebraic information, periodic behavior, and approximation theory all became indispensable in applied problems.

This also explains why Chapter 10 and Chapter 17 belong in the same book. Oscillation, resonance, and series approximations share a history in which mathematical form and physical interpretation kept strengthening one another.

Additional interlude. Gauss, Green, Stokes, and the geometry of fields

By the nineteenth century, calculus was increasingly asked to describe not just curves and moving particles but also fields: gravity, electricity, fluid flow, heat flow, and other quantities spread through regions of space. That shift helps explain why later textbook chapters need divergence, curl, flux, and circulation.

Green's Theorem, the Divergence Theorem, and Stokes' Theorem carry names from different mathematical lives, but together they mark a transition: calculus becomes a language for translating local structure into global measurement in settings where geometry and orientation matter.

The historical point is not merely biographical. It is pedagogical. These theorems entered mathematics because boundary measurements and interior behavior kept appearing together in scientific problems. Once students see that, the theorems stop looking like advanced decorations and start looking like the natural higher-dimensional continuation of the Fundamental Theorem of Calculus.

9. A short note on global mathematical inheritance

The standard story of calculus is often told too narrowly. A more honest picture acknowledges that the mature subject inherited tools from many streams:

- geometric exhaustion and measurement traditions,
- algebraic symbol manipulation,
- trigonometric tables and approximations,
- infinite-process reasoning,
- physical modeling of motion and force,
- numerical procedures developed for practical computation.

That broader inheritance matters for the tone of a textbook. It discourages hero-worship and encourages intellectual humility. Students do not need to memorize a long sequence of names, but they should know that mathematics grows by transmission, revision, and reuse across languages, cultures, and centuries.

10. What the history should change for a student

Historical notes are not included just to make the subject feel literary. They should change how the student reads the mathematics.

Here are the practical takeaways:

- When you meet an integral, remember that its oldest roots are in measurement and approximation.
- When you meet a derivative, remember that tangent, speed, and local prediction problems were historically fused into one idea.
- When a theorem includes hypotheses, read them carefully. They are part of the content, not bureaucratic decoration.
- When late-course vector-calculus theorems appear, look for the local-to-global pattern before memorizing formulas.
- When exact algebra becomes difficult, do not assume the mathematics has failed. Approximation and computation are part of the subject's core identity.

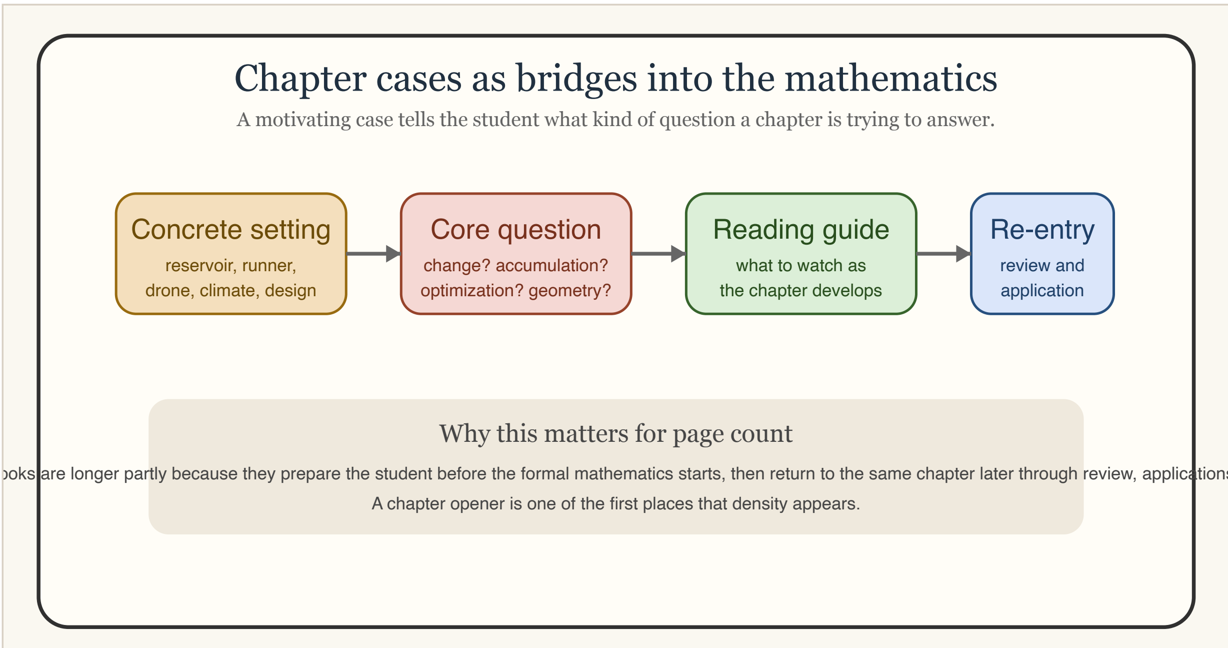
Students often ask whether calculus was "invented" or "discovered." History suggests a better answer: calculus was assembled. It was assembled from problems, methods, notations, and philosophical demands that gradually locked together into the subject we teach now.

Historical reading trail

The appendix was shaped by historical accounts from the MacTutor History of Mathematics Archive, especially entries and topic pages connected to Archimedes, Newton, Leibniz, Euler, Cauchy, Green, Gauss, Stokes, and the method of exhaustion. Additional classroom framing came from the long tradition of calculus-text prefaces that explain the subject through change, accumulation, and approximation rather than through symbol lists alone.

Appendix AD. Chapter-Opening Cases and Reading Guides

Large textbooks often begin chapters with a short motivating case, followed by preview questions that tell students what to watch for. This appendix collects that kind of material in one place. Each case is intentionally brief. The goal is to create a concrete doorway into the chapter, not to bury the reader in pre-chapter exposition.



Chapter 1 case. Tracking a reservoir

A city monitors the volume of water in a reservoir through the week. The total volume matters, but so does how fast it is rising or falling across different time windows.

`Why calculus enters`

The first language of calculus is not the derivative or the integral sign. It is the language of changing quantities and comparison across intervals.

`Reading guide`

- Which quantities vary?
- Which units belong to each?
- What is gained or lost when the data are compressed into one average rate?

Chapter 2 case. A bridge with a sensor glitch

A strain sensor on a bridge reports a smooth rise in load, then briefly loses signal at one instant, then resumes with the same trend.

`Why calculus enters`

This is a natural picture of the difference between function value and nearby behavior. The missing sensor reading does not automatically destroy the local trend.

`Reading guide`

- What does "nearby" mean operationally?
- When does a missing value matter?
- When do left-hand and right-hand behaviors disagree?

Chapter 3 case. Instantaneous speed from GPS data

A runner's app records distance over time. Average pace over a mile is easy to compute, but the coach wants to know how fast the runner was moving at one specific moment during a sprint.

`Why calculus enters`

The derivative grows out of the need to refine average change into instantaneous change.

`Reading guide`

- Why is one interval not enough?
- What does the secant-to-tangent idea repair?
- How does slope become a physical rate?

Chapter 4 case. Chemical concentration depending on nested variables

A concentration depends on temperature, and temperature depends on time through a programmed heating cycle.

`Why calculus enters`

Nested dependence is exactly what the chain rule organizes. The derivative is not one blunt tool. It adapts to structure.

`Reading guide`

- What is inside what?
- Which variable depends on which?
- What gets lost if the inner derivative is ignored?

Chapter 5 case. Designing a package

A manufacturer wants a carton shape that minimizes material while holding a fixed product volume.

`Why calculus enters`

Optimization problems are where derivative interpretation becomes decision-making.

`Reading guide`

- What is the true objective?
- What is fixed and what is free?
- Why is a critical point only part of the answer?

Chapter 6 case. From rainfall rate to flood volume

A storm model gives rainfall intensity in centimeters per hour, but emergency planners need the total volume entering a retention basin across several hours.

`Why calculus enters`

An integral accumulates local contributions into a global total.

`Reading guide`

- What is the local quantity?
- What are the units of a small contribution?
- When would signed accumulation differ from total amount?

Chapter 7 case. Reconstructing motion from velocity

A robot's velocity is known at each moment, and its position at the start of a route is known. The question is how to rebuild the full position function.

`Why calculus enters`

Antiderivatives and initial conditions translate rate data back into state data.

`Reading guide`

- Why do antiderivatives come in families?
- What extra information selects one member?
- Why does differentiating an accumulation function recover the current rate?

Chapter 8 case. Choosing an integration tool under time pressure

An engineering student faces a page of integrals on a placement exam. The main difficulty is not algebra. It is recognizing which method applies before time runs out.

`Why calculus enters`

Technique-heavy chapters are really chapters about diagnosis.

`Reading guide`

- Which forms suggest substitution?
- Which products suggest parts?
- When is a numerical answer the honest answer?

Chapter 9 case. Estimating material for a custom vessel

A shop fabricates a vessel whose walls come from rotating a profile curve around an axis. The shop needs wall area, enclosed volume, and total material mass.

`Why calculus enters`

Area, volume, surface area, and mass all come from local geometry plus accumulation.

`Reading guide`

- What is the representative slice or shell?
- Which units should the final answer have?
- How do different geometric setups describe the same object?

Chapter 10 case. Replacing hard functions by manageable ones

A handheld device needs a fast approximation of e^x near $x = 0$ without computing the full function exactly every time.

`Why calculus enters`

Series chapters explain how local derivative information becomes a practical approximation engine.

`Reading guide`

- What is the difference between a sequence and a series?
- Why can a polynomial approximate a non-polynomial function locally?
- How is the error controlled?

Chapter 11 case. Forecasting a population with limited resources

A wildlife manager models a population that grows quickly when small but slows as habitat pressure increases.

`Why calculus enters`

Differential equations do not only solve for motion. They encode feedback laws.

`Reading guide`

- What does the derivative depend on?
- Where are the equilibria?
- How does a slope field show the model before a formula appears?

Chapter 12 case. Coordinating a drone in space

A drone must pass through three-dimensional waypoints, and the controller tracks both location and velocity vectors.

`Why calculus enters`

Curves in space, velocity, projection, and orientation all demand vector language.

`Reading guide`

- What is a point and what is a vector?
- How does the derivative of a vector function encode motion?
- When do dot and cross products answer different geometric questions?

Chapter 13 case. Reading a temperature landscape

A surface records temperature over a metal plate. The engineer wants to know where temperature rises fastest and how to approximate nearby values.

`Why calculus enters`

Multivariable functions generalize local change from one direction to many directions.

`Reading guide`

- What do contour lines reveal?
- Why is the gradient more than a list of partial derivatives?
- What does the tangent plane approximate?

Chapter 14 case. Total pollutant load in a lake

A pollutant concentration varies across a lake surface and also with depth. Regulators need total load, mean concentration, and likely region-based comparisons.

`Why calculus enters`

Multiple integrals accumulate over areas and volumes, while coordinate choices determine computational cost.

`Reading guide`

- What region or solid is being integrated over?
- Does the geometry suggest rectangular, polar, cylindrical, or spherical coordinates?
- What does the correction factor mean geometrically?

Chapter 15 case. Flow through a turbine

Fluid moves through a turbine housing, and engineers care about circulation in some regions and flux across others.

`Why calculus enters`

Vector calculus organizes quantities distributed through space and across boundaries.

`Reading guide`

- Is the question about travel along a path or flow across a surface?
- Which theorem converts a hard integral into an easier one?
- Why does orientation control sign?

Chapter 16 case. Steering by polar and parametric data

A robotic arm traces a path better described by angle and radial distance than by a single $y = f(x)$ graph.

`Why calculus enters`

Parametric and polar descriptions preserve geometry that rectangular formulas can obscure.

`Reading guide`

- What information is lost when the parameter is eliminated?
- When is a polar picture simpler than a rectangular picture?
- What does curvature say beyond slope?

Chapter 17 case. Vibration in a suspension system

A suspension component is displaced, released, and then driven by periodic road forcing while damping removes energy.

`Why calculus enters`

Second-order equations describe restoring force, damping, and resonance in a unified framework.

`Reading guide`

- What do the characteristic roots predict?
- When does oscillation decay?
- Why can forcing produce large responses near resonance?

Chapter 18 case. Long-tail risk

A risk analyst studies rare events whose probability density stretches across a long domain. The chance of extreme outcomes is small, but the total tail contribution still matters.

`Why calculus enters`

Improper integrals measure whether an infinite or singular accumulation remains finite.

`Reading guide`

- What is the benchmark comparison function?
- Where is the potential divergence: at infinity or at a singular point?
- How does continuous tail behavior connect with discrete series behavior?

Closing note

One reason long textbooks feel substantial is that they do not assume the reader already sees why a chapter matters. They build that bridge explicitly. This appendix is intended to serve that role: a doorway into the mathematics before the formal development begins.

Appendix U. Theory, Proof, and Modeling Extension Bank

This appendix adds problems that sit between routine exercises and full projects. They are intended for honors sections, writing assignments, take-home quizzes, discussion sections, or any course that wants the manuscript to behave more like a large commercial text with theory, explanation, and modeling prompts throughout.

Suggested uses

- Assign one prompt each week as a writing exercise.
- Use proof-oriented prompts in recitation or honors support sections.
- Use modeling prompts as the bridge between computational technique and real interpretation.

Chapter 1

1. Write a one-page note explaining why units are not decorative in rate problems.
2. Give two different real situations that produce the same linear formula and explain what the same slope means differently in each context.
3. Construct a quantity that has constant average rate of change on two intervals but not on the union of those intervals.
4. Explain why local linearity is already a mathematical approximation principle, even before derivatives are formally defined.

Chapter 2

1. Write a paragraph explaining why continuity is a statement about behavior, not only about formulas.
2. Give an example where left-hand and right-hand limits exist but differ, then explain what the graph is doing.
3. Explain why a multivariable path test can disprove a limit but cannot by itself prove one.
4. Write a short note connecting epsilon-delta language to the informal phrase "arbitrarily close."

Chapter 3

1. Show from the definition that the derivative of a constant function is zero.
2. Explain why continuity is necessary but not sufficient for differentiability.
3. Describe how the secant-to-tangent idea is an approximation argument.
4. Write a short paragraph comparing derivative as slope and derivative as rate.

Chapter 4

1. Explain why the product rule cannot be replaced by differentiating each factor independently and multiplying.
2. Give a physical or geometric meaning for the chain rule as nested dependence.
3. Write a short proof sketch for the derivative of e^x assuming it equals its own derivative.
4. Compare implicit differentiation and explicit differentiation as modeling tools.

Chapter 5

1. Write a sign-chart argument showing why $f'(x) = 0$ at $x = 0$ for $f(x) = x^3$ does not produce a local extremum.
2. Explain the difference between a turning point and an inflection point.
3. Build an optimization problem where algebra gives several critical points but context rules out at least one of them.
4. Describe why concavity information improves graphing beyond slope alone.

Chapter 6

1. Explain why a definite integral is a limit of sums without copying a formal theorem statement.
2. Compare net accumulation and total accumulation using one physical and one financial example.
3. Write a paragraph showing how finite additivity is reflected in the integral over adjacent intervals.
4. Explain why average value over an interval is not the same thing as sampling the midpoint once.

Chapter 7

1. Write a short proof sketch of why all antiderivatives of the same function differ by a constant.
2. Explain the Fundamental Theorem of Calculus as a bridge between local rate and total change.
3. Give an example where substitution is obvious only after rewriting the integrand.
4. Describe how initial-value problems turn antiderivative families into specific solutions.

Chapter 8

1. Explain why method selection is a conceptual issue and not only a computational issue in integration.
2. Compare substitution and integration by parts using the language of structural recognition.
3. Give a situation in which a numerical approximation is mathematically preferable to a difficult exact antiderivative.
4. Write a short note describing how partial fractions is really an algebra problem inside a calculus problem.

Chapter 9

1. Explain why the choice of slice or shell happens before integration, not during it.
2. Compare work, mass, and hydrostatic force as three variants of weighted accumulation.
3. Give a units-based argument showing why an incorrect integrand can often be detected before any computation.
4. Explain how the area/volume foundations appendix changes the way you read application formulas in this chapter.

Chapter 10

1. Write a paragraph explaining why the harmonic series diverges even though its terms go to zero.
2. Compare absolute and conditional convergence in a way that a first-time student could understand.
3. Explain why power series are not just formal algebraic gadgets but computational tools.
4. Describe the role of remainder estimates in making Taylor approximations trustworthy.

Chapter 11

1. Explain why a slope field can reveal long-term behavior before any exact solution is written.
2. Compare exponential and logistic growth models in terms of the assumptions built into the rate law.
3. Write a short note distinguishing equilibrium analysis from exact solution.
4. Explain how a numerical method can be correct as an algorithm but still poor as an approximation if step size is chosen badly.

Chapter 12

1. Explain why a direction vector and a normal vector answer different geometric questions.
2. Compare speed and velocity in a motion model that changes direction but not speed.
3. Show how the cross product encodes both magnitude of area and orientation.
4. Write a short paragraph on why parametrized curves are natural in motion problems.

Chapter 13

1. Explain why a gradient is a derivative object and a geometric object at the same time.
2. Compare tangent lines from single-variable calculus with tangent planes in several variables.
3. Write a short note explaining why saddle points are central in multivariable optimization.
4. Explain why Lagrange multipliers fit the language of constrained movement along a surface or curve.

Chapter 14

1. Explain why changing order of integration is often a geometry problem disguised as an algebra problem.
2. Give a local-area explanation for the polar Jacobian factor `r`.
3. Compare a triple integral for mass with a triple integral for probability.
4. Write a paragraph describing how symmetry simplifies multiple integration.

Chapter 15

1. Explain why the major integral theorems in vector calculus resemble the Fundamental Theorem of Calculus.
2. Compare circulation and flux without writing formulas.
3. Give a short note on why conservative fields reduce line integrals to endpoint data.
4. Explain how orientation mistakes can reverse the meaning of an answer in vector calculus.

Chapter 16

1. Explain why a curve may be easy to parametrize but hard to describe explicitly as $y = f(x)$.
2. Compare rectangular, parametric, and polar descriptions of the same circle.
3. Write a geometric explanation of why polar area scales like r^2.
4. Explain how curvature improves on slope as a local descriptor of shape.

Chapter 17

1. Explain why repeated roots produce threshold behavior between nonoscillation and oscillation.
2. Compare simple harmonic motion with damped oscillation in terms of energy.
3. Write a short note explaining why resonance is a mathematical idea with real engineering consequences.
4. Contrast initial-value and boundary-value problems using one mechanical example and one geometric example.

Chapter 18

1. Explain why improper integrals should be thought of as convergence questions rather than ordinary area formulas with unusual notation.
2. Compare the tails of `1/x`, $1/x^2$, and e^{-x} in a way that emphasizes accumulation rather than only pointwise decay.
3. Write a short note describing how the integral test turns a series problem into a continuous accumulation problem.
4. Explain why improper integrals are central in probability even when the support of a density is infinite.

Cross-chapter synthesis prompts

1. Write an essay connecting local linearity, Euler stepping, and tangent-based approximation across Chapters 1, 4, 11, and 17.
2. Compare the role of coordinate choice in Chapters 12, 14, and 16.
3. Explain how accumulation ideas connect Chapters 6, 7, 9, 14, and 15.
4. Describe how derivatives evolve from slope in Chapter 3 to gradient in Chapter 13 and to divergence/curl in Chapter 15.

Appendix A. Prerequisite Review

This appendix is not a substitute for a full algebra or trigonometry course. It is a compact review of the ideas that repeatedly support successful calculus work.

A.1 Algebraic structure

Calculus uses algebra constantly. The difference is that algebra is no longer the final goal; it is the language needed to reveal calculus structure.

Key habits:

- factor before cancelling,
- distribute carefully and then collect like terms,
- move between fractional and exponent forms without changing meaning,
- and check whether an expression is defined before manipulating it.

Common algebra moves that matter in calculus

- Difference of squares: $a^2 - b^2 = (a - b)(a + b)$
- Perfect-square trinomials: $a^2 + 2ab + b^2 = (a + b)^2$
- Common-factor extraction: $ab + ac = a(b + c)$
- Rational simplification: cancel factors, not terms

When a limit problem gives $0/0$, the right response is often algebraic simplification, not panic.

A.2 Exponents and logarithms

The derivative and integral chapters involving exponentials and logarithms depend on a fluent command of basic identities.

Useful rules:

- $a^m a^n = a^{m+n}$
- $a^m / a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$

- $(ab)^n = a^n b^n$
- $\log(ab) = \log a + \log b$
- $\log(a/b) = \log a - \log b$
- $\log(a^p) = p \log a$

These rules are not optional decoration. They often determine whether a derivative or integral is visible in a workable form.

A.3 Function language

Before calculus asks how a function changes, you need a stable sense of what a function is.

Important vocabulary:

- `input`: the allowed independent quantity,
- `output`: the resulting dependent quantity,
- `domain`: the allowed inputs,
- `range`: the outputs that actually occur,
- `composition`: one function fed into another,
- `inverse`: a function that reverses another function when the reversal makes sense.

Whenever possible, describe a function in words as well as symbols.

A.4 Graph interpretation

Calculus relies on reading graphs intelligently. That means more than identifying intercepts.

Look for:

- intervals of increase and decrease,
- steepness,
- symmetry,
- intercepts,
- asymptotic behavior,
- local highs and lows,
- and how the graph changes shape.

A graph never replaces algebra entirely, but it often prevents algebra from being interpreted badly.

A.5 Trigonometry essentials

Single-variable and multivariable calculus both assume basic trigonometric fluency.

Core facts:

- $\sin^2 x + \cos^2 x = 1$
- right-triangle definitions of sine, cosine, and tangent,
- the unit-circle interpretation of trigonometric functions,
- inverse trigonometric functions as angle-recovery tools,
- and the radian measure of angle.

Radian measure matters especially in calculus because the clean derivative formulas for sine and cosine depend on it.

A.6 Coordinate geometry

You should be comfortable with:

- slope between two points,
- distance in the plane,
- midpoint,
- equation of a line,
- basic conic and polynomial graph shapes,
- and the meaning of coordinates in two and three dimensions.

These geometric tools reappear in tangent lines, local linearization, optimization, and multivariable visualization.

A.7 Units and dimensional thinking

Dimensional thinking is one of the fastest ways to catch mistakes.

Examples:

- a derivative of position with respect to time should be a length-per-time quantity,
- an integral of density times length should produce mass,
- a double integral of surface density over area should produce total mass.

If the units fail, the setup is probably wrong.

A.8 Scientific notation and scale

Applied calculus often works across large and small scales.

Review:

- 3.2×10^5 means move the decimal five places to the right,
- 4.7×10^-3 means move the decimal three places to the left,
- and relative size matters as much as exact decimal expansion.

Scale awareness becomes especially important in modeling, error analysis, and numerical work.

A.9 Productive review habits

If a prerequisite idea feels shaky, do not wait for the chapter to fail. Repair it early.

A reliable review cycle is:

1. work one plain algebra or trig example,
2. explain the rule in words,
3. use it immediately inside a calculus problem,
4. and then check whether the calculus meaning still makes sense.

A.10 Self-check list

Before pushing deep into the main text, you should be able to do the following without heavy hesitation:

- simplify $(x^2 - 9)/(x - 3)$ when $x \neq 3$,
- interpret a function from a table and from a graph,
- compute slope between two points,
- solve a basic linear equation,
- use unit-circle values of sine and cosine at standard angles,
- and explain what units a rate of change should have in a simple context.

If several of those still feel unstable, use this appendix as a repair guide while reading the main chapters.

Appendix B. Formula and Pattern Reference

This appendix is a compact reference, not a replacement for understanding. Use it to refresh memory after the ideas themselves are in place.

B.1 Difference quotients and local linearity

- Average rate of change on `[a, b]`:

$$(f(b) - f(a))/(b - a)$$

- Derivative at `a`:

$$f'(a) = \lim_{h \to 0}(f(a + h) - f(a))/h$$

- Linearization at $x = a$:

$$L(x) = f(a) + f'(a)(x - a)$$

B.2 Core derivative patterns

- $d/dx(c) = 0$
- $d/dx(x^n) = nx^{n-1}$
- $d/dx(e^x) = e^x$
- $d/dx(a^x) = a^x \ln a$
- $d/dx(\ln x) = 1/x$
- $d/dx(\sin x) = \cos x$
- $d/dx(\cos x) = -\sin x$

B.3 Derivative structure rules

- Product rule:

$$(fg)' = f'g + fg'$$

- Quotient rule:

$$(f/g)' = (f'g - fg')/g^2$$

- Chain rule:

$$(f(g(x))) = f\ (g(x))g\ (x)$$

- Implicit differentiation: differentiate both sides with respect to the active variable and solve for the desired derivative

B.4 Meaning cues for derivatives

- positive derivative: local increase
- negative derivative: local decrease
- zero derivative: horizontal tangent candidate
- positive second derivative: concave up
- negative second derivative: concave down

B.5 Core antiderivative patterns

- $int x^n dx = x^{n+1}/(n+1) + C$, for $n \neq -1$
- $int 1/x dx = \ln|x| + C$
- $int e^x dx = e^x + C$
- $int \sin \times dx = -\cos x + C$
- $int \cos \times dx = \sin x + C$
- $int \sec^2 \times dx = \tan x + C$

B.6 Definite integral facts

- $\int_a^b f(x)dx$ is accumulated total or net change
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- $\int_b^a f(x)dx = -\int_a^b f(x)dx$
- average value on `[a, b]`:

$$(1/(b - a)) \int_a^b f(x)dx$$

B.7 Integration-method cues

- `substitution`: use when part of the integrand appears together with its inside derivative
- `integration by parts`: use when the integrand is a product and differentiating one factor simplifies it

- `partial fractions`: use for rational functions after denominator factoring
- `trigonometric substitution`: use when square roots resemble $a^2 - x^2$, $a^2 + x^2$, or $x^2 - a^2$
- `numerical integration`: use when data are tabulated or an exact antiderivative is not practical

B.8 Geometric application templates

- area between curves:

$$\int_a^b (\text{top} - \text{bottom})dx$$

- volume by slices:

$$\int_a^b A(x)dx$$

- volume by shells:

$$\int_a^b 2\text{pi}(\text{radius})(\text{height})dx$$

- work:

$$\int_a^b F(x)dx$$

- mass of a rod:

$$\int_a^b \varrho(x)dx$$

B.9 Sequences and series cues

- sequence convergence: terms approach one limit
- series convergence: partial sums approach one limit
- geometric series: $a + ar + ar^2 + \ldots = a/(1 - r)$ when $|r| < 1$
- Taylor polynomial: local polynomial built from derivatives at a point

B.10 Differential-equation cues

- separable form: gather `y` terms with dy and `x` terms with dx
- exponential growth/decay:

$$y = ky$$

- logistic growth:

$$P = kP(1 - P/M)$$

- Euler's method: next value = current value + slope times step size

B.11 Multivariable and vector formulas

- gradient:

$$\nabla f = \langle f_x, f_y \rangle$$

- tangent-plane approximation:

$$f(a,b) + f_x(a,b)(x - a) + f_y(a,b)(y - b)$$

- double integral over a rectangle:

$$\int_a^b \int_c^d f(x,y)dydx$$

- polar area element:

$$dA = r dr dtheta$$

- dot product:

$$u \cdot v = |u||v| \cos \theta$$

- cross product magnitude:

$$|u \times v| = |u||v| \sin \theta$$

B.12 The most durable pattern in the whole book

Again and again, calculus works by the same strategy:

1. identify a small local contribution,
2. express it with the right units,
3. combine the pieces carefully,
4. and interpret the result back in the original situation.

That pattern outlasts any single formula list.

Appendix C. Glossary

This glossary is intentionally compact. Definitions are written for fast recall, not as replacements for the fuller chapter discussions.

A

- `acceleration`: the rate of change of velocity.
- `accumulation`: a total formed by adding many small contributions.
- `antiderivative`: a function whose derivative is the given function.
- `average rate of change`: output change divided by input change over an interval.
- `average value`: the constant output that would produce the same total accumulation on an interval.

B

- `boundary`: the edge of a region, interval, curve, or surface.

C

- `chain rule`: the derivative rule for compositions of functions.
- `concave down`: a graph whose slopes tend to decrease as you move left to right.
- `concave up`: a graph whose slopes tend to increase as you move left to right.
- `continuity`: agreement between nearby behavior and actual function value at a point.
- `critical point`: a point where $f' = 0$ or where `f'` does not exist.

D

- `definite integral`: the accumulated total defined as a limit of Riemann sums.
- `delta`: the input tolerance in the formal language of limits.
- `density`: amount per unit length, area, or volume.
- `derivative`: the instantaneous rate of change or tangent slope.
- `differentiable`: having a derivative.
- `difference quotient`: a ratio of output change to input change.

- `directional change`: change measured in a chosen direction in multivariable settings.
- `divergence`: a measure of source-like or sink-like behavior in a vector field.
- `domain`: the set of allowed inputs of a function.

E

- `epsilon`: the output tolerance in the formal language of limits.
- `equilibrium solution`: a constant solution to a differential equation.
- `Euler's Method`: a step-by-step numerical method for approximating solutions of differential equations.

F

- `flux`: the net flow crossing a surface or curve.
- `Fundamental Theorem of Calculus`: the theorem linking derivatives and definite integrals.
- `function`: a rule that assigns one output to each allowed input.

G

- `geometric series`: a series with a constant ratio between successive terms.
- `gradient`: the vector of partial derivatives pointing in the direction of steepest increase.
- `Green's Theorem`: a theorem relating a planar boundary integral to an interior double integral.

H

- `higher derivative`: a derivative of a derivative, such as the second derivative.

I

- `implicit differentiation`: differentiation applied to an equation not solved explicitly for one variable.
- `indeterminate form`: an expression such as $0/0$ that requires more analysis before a limit can be evaluated.
- `independent variable`: the input quantity in a function relationship.

- `inflection point`: a point where concavity changes.
- `initial value problem`: a differential equation plus a condition that selects one specific solution.
- `integral`: an accumulation operator or accumulated quantity depending on context.

L

- `left-hand limit`: the nearby behavior of a function when approached from smaller input values.
- `limit`: the value a function approaches as the input approaches a target.
- `linear approximation`: the tangent-line-based local model of a function.
- `local extremum`: a nearby maximum or minimum.
- `local linearity`: the idea that many functions look nearly linear when viewed closely enough.
- `logistic model`: a growth model with a limiting carrying capacity.

M

- `marginal`: another word for derivative in many economic settings.
- `model`: a mathematical representation of a real system.

N

- `net change`: the final accumulated change after positive and negative contributions are combined.
- `nondifferentiable`: lacking a derivative at the point under discussion.

O

- `optimization`: the process of finding best possible values under given conditions.

P

- `parametric curve`: a curve described by coordinate functions of a parameter.
- `partial derivative`: the rate of change with respect to one variable while others are held fixed.
- `partial fractions`: a decomposition of rational functions into simpler pieces.

- `path dependence`: the failure of a multivariable limit to have the same value along different approaches.
- `power series`: an infinite polynomial centered at a point.
- `product rule`: the derivative rule for products.

Q

- `quotient rule`: the derivative rule for quotients.

R

- `radius of convergence`: the distance from the center within which a power series converges.
- `rate`: change per unit of input.
- Riemannsum: a sum of small products approximating accumulated total.
- `right-hand limit`: the nearby behavior of a function when approached from larger input values.

S

- `secant line`: a line through two points of a curve.
- `separable equation`: a differential equation that can be reorganized so variables separate.
- `series`: an infinite sum defined through its partial sums.
- `shell method`: a volume method using cylindrical shells.
- `signed area`: area counted positively above an axis and negatively below it.
- `slope field`: a diagram of local slopes for a differential equation.
- `Stokes' Theorem`: a theorem relating circulation on a boundary to rotational behavior across a surface.
- `substitution`: an integration method that reverses the chain rule.

T

- `tangent line`: the best local linear fit to a curve at a point.
- `tangent plane`: the best local linear fit to a surface at a point.
- `Taylor polynomial`: a local polynomial built from derivative information at a point.
- `trapezoidal rule`: a numerical integration method using trapezoids.

V

- `vector`: a quantity with magnitude and direction.
- `vector field`: a vector assigned to each point in a region.
- `velocity`: the rate of change of position.
- `volume element`: a small three-dimensional piece used in triple integrals.

W

- `work`: accumulated force along displacement.

Appendix S. Foundations of Area and Volume

This appendix records an important point that many calculus books leave implicit:

- area is a function,
- volume is a function,
- each has a domain,
- and each is characterized by a small set of geometric properties.

The integral later extends these ideas, but it does not create them from nothing. Before an integral can measure area or volume in familiar situations, there must already be a geometric notion of area and volume for simple regions and solids.

S.1 Area as a finitely additive function

Think of area as a function

$$A(region)$$

defined first on a class of bounded planar regions for which geometric area makes sense. A practical introductory domain is:

- polygons and finite unions of polygons,
- and then, more generally, bounded Jordan-measurable planar sets.

The phrase "Jordan measurable" means, roughly, that the set can be approximated from inside and outside by finite unions of rectangles whose areas can be made arbitrarily close.

Core properties of area

The standard geometric properties are:

1. $A(unitsquare) = 1$.
2. $A(R) \geq 0$ for every region `R` in the domain.
3. Congruent regions have the same area.
4. Finite additivity: if $R_1, ..., R_n$ have pairwise disjoint interiors and their union is still in the domain, then

$A(R_1 union ... union R_n) = A(R_1) + ... + A(R_n)$.

These four properties already force much of elementary geometry.

Why finite additivity matters so much

Finite additivity is the real engine behind geometric formulas. It says that if a complicated region can be cut into finitely many simpler pieces, then the total area is the sum of the areas of those pieces.

Without that principle, formulas for rectangles, triangles, polygons, and later integral-based regions would have no structural glue.

Monotonicity follows from additivity

If `R subset S` and both lie in the domain, then

$$A(S) = A(R) + A(S R) \geq A(R).$$

So larger regions have at least as much area as smaller ones. This property is usually treated as obvious, but in fact it comes from nonnegativity plus additivity.

S.2 From the unit square to familiar formulas

Square of side `s`

Start with the unit square, whose area is `1`.

If a square has side length `n`, where `n` is a positive integer, then it can be partitioned into n^2 unit squares. Finite additivity gives

$$A(square of side n) = n^2.$$

If the side length is `1/n`, then n^2 such small squares tile the unit square, so

$$A(square of side 1/n) = 1/n^2.$$

More generally, if the side length is a rational number `m/n`, then the area is

$$(m/n)^2.$$

To extend from rational side lengths to arbitrary real side lengths, one uses a regularity or approximation principle: squeeze the square between rational-sided squares from below and above. That step is exactly the kind of limiting idea that later reappears in integration.

So the natural formula is

$$A(square of side s) = s^2.$$

Rectangle of side lengths `a` and `b`

Once squares are understood, rectangles follow by the same subdivision idea:

$A(\text{rectangle}) = ab.$

For integer or rational side lengths this is immediate from tiling by smaller congruent squares. For general real side lengths, approximation fills the gap.

Triangle of base `b` and height `h`

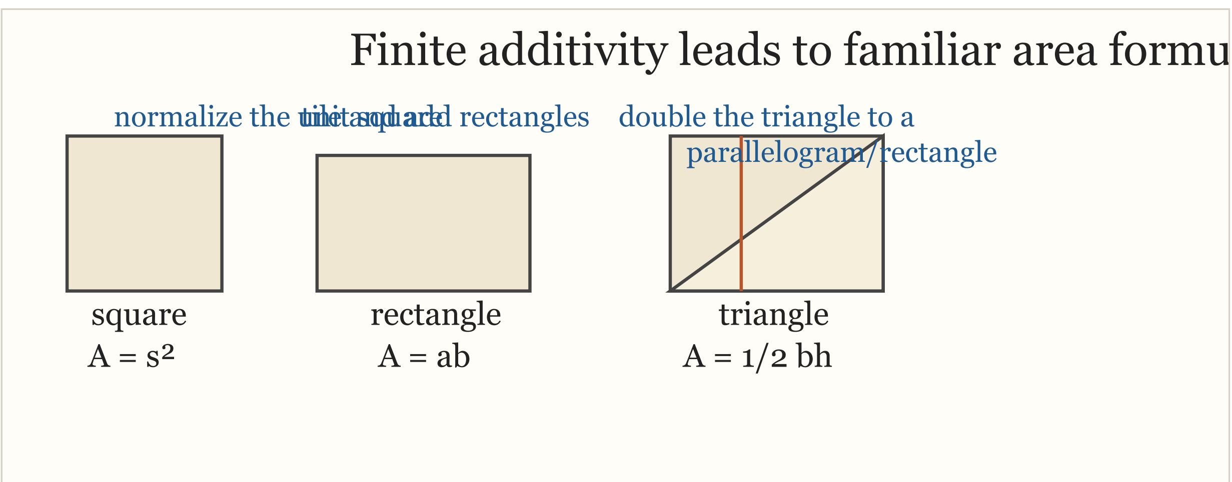
Any triangle can be paired with a congruent copy to form a parallelogram with base `b` and height `h`. A parallelogram, in turn, can be cut and rearranged into a rectangle of the same base and height, so its area is `bh`.

Therefore the triangle occupies half of that:

$A(\text{triangle}) = 1/2bh.$

This formula is not a new axiom. It is a consequence of:

- congruence,
- finite additivity,
- and the rectangle formula.



Why this is already a calculus idea

The move from unit square to rectangle to triangle is a prototype of later calculus reasoning:

- define a simple standard object,
- decompose more complicated objects into standard pieces,
- preserve the quantity under congruence,
- and use limiting arguments when exact tilings are replaced by approximations.

S.3 Volume as a finitely additive function

Now think of volume as a function

$$V(\text{solid})$$

defined first on a class of bounded solids such as:

- boxes,
- prisms,
- polyhedra,
- and then, more generally, bounded Jordan-measurable solids.

Core properties of volume

The parallel axioms are:

1. $V(\text{unit cube}) = 1$.
2. $V(S) \geq 0$ for every solid `S` in the domain.
3. Congruent solids have the same volume.
4. Finite additivity: if $S_1, \dots, S_n$ have pairwise disjoint interiors and their union is in the domain, then

$V(S_1 \text{ union} \dots \text{union } S_n) = V(S_1) + \dots + V(S_n)$.

These properties play the same role in space that the area axioms play in the plane.

Cube of side `s`

Exactly as with squares,

$V(\text{cube of side } s) = s^3$.

Rectangular box of side lengths `a`, `b`, and `c`

Subdivide the box into cubes or compare it with repeated stacking of base rectangles. The standard formula is

$V(\text{box}) = abc$.

Prism of base area `B` and height `h`

If a solid is a prism with congruent cross-sections stacked straight upward, then

$V(\text{prism}) = Bh$.

This is the three-dimensional version of rectangle area.

S.4 Where the $1/3$ comes from for a tetrahedron

The tetrahedron plays, in three dimensions, a role somewhat analogous to the triangle in two dimensions. But the coefficient is not $1/2$; it is $1/3$.

Why?

A coordinate tetrahedron inside a box

Consider the tetrahedron bounded by the coordinate planes and the plane through `(a,0,0)`, `(0,b,0)`, and `(0,0,c)`.

Its base in the `xy`-plane is a right triangle of area

$B = 1/2ab,$

and its height is `c`.

Now place this tetrahedron inside the rectangular box with side lengths `a`, `b`, and `c`. That box has volume

`abc`.

The box can be decomposed into six congruent tetrahedra of this type, so each tetrahedron has volume

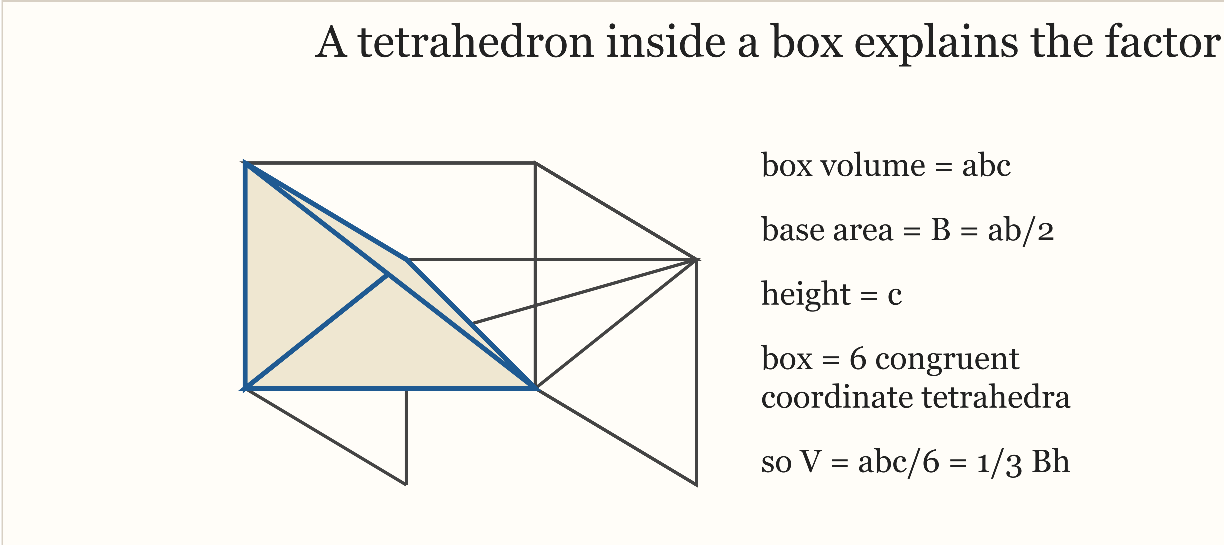
`abc/6`.

But

$abc/6 = 1/3(ab/2)c = 1/3Bh.$

So for this tetrahedron,

$V = 1/3Bh.$



From this special case to general pyramids and tetrahedra

There are several ways to justify the general formula:

- by affine geometry,
- by Cavalieri's principle,
- or by slicing.

In a calculus setting, slicing gives the cleanest explanation. If a pyramid has base area `B` and height `h`, then a slice at height `x` above the apex is similar to the base, scaled by the factor `x/h`. Its area is therefore proportional to $(x/h)^2$.

Measured from the apex downward, the cross-sectional area takes the form

$$A(x) = B(x/h)^2.$$

So the volume is

$$\int_0^h B(x/h)^2 dx = (B/h^2)\int_0^h x^2 dx = (B/h^2)(h^3/3) = 1/3Bh.$$

Thus the coefficient $1/3$ comes from the integral of a square-scaling law.

Triangle versus tetrahedron

This comparison is worth saying explicitly:

- a triangle has half the area of the matching parallelogram or rectangle,
- a tetrahedron or pyramid has one third the volume of the matching prism with the same base area and height.

The factors differ because one situation is built from linear scaling in one perpendicular direction, while the other is built from cross-sections whose area scales quadratically.

S.5 Jordan area, Jordan volume, and why some sets fail

For ordinary geometric regions, the axioms above work well. But if the boundary of a set is too wild, area or volume may fail to exist in the Jordan sense.

Inner and outer approximation

For a bounded set `E` in the plane or in space:

- the inner Jordan content comes from fitting finite unions of rectangles or boxes inside `E`,
- the outer Jordan content comes from covering `E` by finite unions of rectangles or boxes.

If the inner and outer values agree, the set has Jordan area or Jordan volume.

A bounded set with no Jordan area

Take all rational points inside the unit square.

This set has no Jordan area.

Why?

- No nontrivial rectangle lies entirely inside it, because every rectangle contains irrational points. So the inner Jordan area is `0`.
- But the rational points are dense in the unit square. Any finite union of rectangles that covers all rational points in the square must actually cover the whole square. So the outer Jordan area is `1`.

Since inner and outer values do not agree, the set has no Jordan area.

A bounded set with no Jordan volume

The same idea works in three dimensions: the rational points inside the unit cube have inner Jordan volume `0` and outer Jordan volume `1`, so they have no Jordan volume.

Why this matters in a calculus book

Most sets that arise in beginning calculus are perfectly well behaved:

- intervals,
- polygons,

- disks,
- regions under graphs,
- solids cut by smooth surfaces.

So the failure of area or volume is not an everyday obstacle. But it is conceptually important because it shows that "bounded subset of space" is not, by itself, a sufficient domain for area or volume.

S.6 What this appendix changes about earlier chapters

This appendix sharpens several statements from the main text.

When Chapter 6 uses area pictures for integrals, the picture is not supposed to be the definition of the integral. Rather:

- area already exists for simple geometric regions,
- signed accumulation extends and reorganizes that idea,
- and the integral gives a far more flexible quantity than geometric area alone.

When Chapter 9 discusses volume by slices, it is building on:

- volume as a finitely additive set function,
- standard prism and box formulas,
- and the fact that limiting or slicing arguments can justify new formulas from old ones.

So the integral is best viewed not as replacing geometry, but as extending geometry to regions and solids whose measurement is naturally organized by continuously varying slices.

S.7 Exercises and writing prompts

1. Explain why finite additivity implies monotonicity of area.
2. Give a dissection argument for the area formula $A(\text{rectangle}) = ab$ starting from the unit square.
3. Explain why the triangle formula depends on congruence invariance as well as additivity.
4. Show that if a cube of side `n` is tiled by unit cubes, then its volume must be n^3.
5. Explain why a prism with base area `B` and height `h` should have volume `Bh`.
6. In your own words, where does the factor $1/3$ in a pyramid or tetrahedron volume formula come from?
7. Why do the rational points in the unit square fail to have Jordan area?

8. Write a short paragraph comparing the role of approximation in extending area from rational-sided rectangles to arbitrary rectangles with the role of approximation in defining the definite integral.

Appendix W. Formula Derivation Notes

This appendix gathers short derivation notes for formulas that students often memorize too early. The goal is not to replace rigorous proof, but to keep the main geometric or analytic idea visible. Large calculus texts often spread this kind of material through sidebars and theorem boxes; here it is collected in one place for review and classroom reference.

W.1 Difference quotients and tangent slope

The derivative formula

$$f\ (a) = \lim_{(} h \to 0)(f(a+h) - f(a))/h$$

comes from the slope of secant lines. The numerator measures output change. The denominator measures input change. For a fixed nonzero `h`, the quotient is an average rate of change over an interval. Letting `h` shrink asks whether these interval slopes settle to a unique local slope. This is the conceptual reason the derivative belongs to both geometry and motion.

W.2 Product and chain rules

The product rule can be motivated by looking at

$(f + \Delta f)(g + \Delta g) - fg = f\Delta g + g\Delta f + \Delta f\Delta g.$

After dividing by the input change and taking the limit, the last term becomes negligible because it is second order. That leaves `f g' + g f'`.

The chain rule is an organized statement about nested dependence. If `y` depends on `u` and `u` depends on `x`, then a small change in `x` affects `y` through the intermediate change in `u`. The derivative multiplies these rates because the two change processes are linked in series, not in parallel.

W.3 Linearization

The linearization

$$L(x) = f(a) + f'(a)(x - a)$$

is the tangent line written as a local model rather than as a geometric ornament. It is the best first-order approximation because it matches both value and slope at the anchor point. The reason it works is local linearity: on a short interval, a differentiable function behaves increasingly like its tangent line.

W.4 Riemann sums and the definite integral

A Riemann sum breaks an interval into many small subintervals, chooses a representative height on each one, and adds the areas of the resulting rectangles. If the function is well behaved and the subinterval widths go to zero, these approximations settle to a single number. That limiting number is the definite integral. The formula is therefore not primarily about antiderivatives. It is about stable accumulation from finer and finer local approximations.

W.5 The Fundamental Theorem of Calculus

The Fundamental Theorem links two questions that at first seem unrelated:

- how fast is an accumulation function changing,
- and how can a total accumulation be found from an antiderivative?

If $A(x) = \int_a^x f(t)dt$, then over a short interval the extra accumulation is approximately $f(x)$ times the interval width. Dividing by the width suggests $A'(x) = f(x)$. That is the local half of the theorem. The evaluation half follows by asking for any function whose derivative is `f`: if $F' = f$, then the total accumulation from `a` to `b` must be $F(b) - F(a)$.

W.6 Substitution and integration by parts

Substitution is the reverse side of the chain rule. When an integrand looks like $f(g(x))g'(x)$, the inner structure $g(x)$ is the natural new variable because differentiation of the inside already appears in the integrand.

Integration by parts is the reverse side of the product rule:

$(uv)' = u'v + uv'.$

Integrating and rearranging gives

$intudv = uv - intvdu.$

So both methods are not arbitrary tricks. They are inverse-derivative principles tied to the structure of earlier differentiation rules.

W.7 Arc length and surface area

Arc length comes from approximating a curve by many tiny line segments. If a graph changes by dx horizontally and dy vertically, the local length is approximately

$\sqrt{dx^2 + dy^2}$.

Substituting $dy = f'(x)dx$ gives the familiar integrand $sqrt(1 + (f')^2)$.

Surface area of revolution uses the same local idea one level up. A tiny arc segment of length `ds` swept around an axis creates a narrow ribbon whose area is approximately circumference times slant length. That is why formulas involve a radius factor together with `ds`.

W.8 Polar area and Jacobian factors

The polar area element is not a mysterious correction factor. A thin polar sector with radius `r` and angle `dtheta` has arc length approximately `r dtheta`. The sector area is therefore approximately

$$(1/2)r(r dtheta) = (1/2)r^2 dtheta$$

for single-curve area, or `r dr dtheta` for double-integral area accumulation over a tiny polar rectangle. The extra `r` appears because equal changes in angle sweep larger arcs farther from the origin. Jacobian factors in other coordinate changes serve the same purpose: they account for geometric distortion of local area or volume.

W.9 Comparison tests and the integral test

Comparison methods work because convergence is often controlled by the largest-scale tail behavior rather than by local decorative algebra. If a function is trapped below a known convergent benchmark, its total accumulation cannot exceed a finite quantity. If it stays above a known divergent benchmark, divergence is unavoidable.

The integral test is the series version of the same principle. A positive decreasing sequence defines a staircase. The nearby integral compares that staircase to an area whose tail is easier to analyze continuously. Convergence of one reflects convergence of the other.

W.10 Euler stepping and numerical modeling

Euler's Method is not merely a numerical recipe. It is the direct computational form of local linearity. At each step:

- use the current derivative,
- assume the function behaves approximately linearly for a short time,
- update the value,
- and repeat.

The method works better with smaller steps because the tangent-line approximation is trusted only over short intervals. Improved methods succeed by capturing more slope information per step, but the governing idea is the same.

W.11 Gradient, tangent plane, and Lagrange multiplier logic

The gradient appears because it is the multivariable object that packages all first-order directional change into one vector. A tangent plane is the linear approximation built from that first-order data, just as a tangent line is the one-variable linear approximation.

Lagrange multipliers arise when optimization is restricted to a constraint curve or surface. At an extremum, the direction of greatest increase of the objective cannot point along an allowed direction of motion on the constraint. So the objective gradient must line up with the constraint gradient. That is the geometric origin of $\nabla f = \lambda \nabla g$.

W.12 Flux, circulation, and theorem families

The major theorems of vector calculus are not unrelated formulas. They all share one pattern:

- local behavior inside a region,
- global accumulation on the boundary.

The Fundamental Theorem of Calculus compares derivative information on an interval with function values at endpoints. Green's Theorem compares local rotation or divergence-like information in a region with boundary circulation or flux. Stokes' Theorem lifts the same pattern to surfaces and their boundary curves. The Divergence Theorem compares interior source strength with flux through the enclosing surface.

Seeing these as one family makes the formulas easier to remember and harder to misuse.

How to study with this appendix

- Read the relevant note before attempting a difficult homework set.
- Turn each note into a spoken explanation without looking at the page.
- Compare each derivation note with a worked example from the matching chapter.
- Use the notes to decide whether you are memorizing a formula or actually understanding where it comes from.

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- `assets/graphics/average-vs-local-rate.svg`
- `assets/graphics/accumulation-strips.svg`
- `assets/graphics/signed-vs-total-accumulation.svg`
- `assets/graphics/limit-hole-and-jump.svg`
- `assets/graphics/limit-neighborhood-bands.svg`
- `assets/graphics/secant-to-tangent.svg`
- `assets/graphics/position-velocity-acceleration-map.svg`
- `assets/graphics/modern-calculus-uses-map.svg`
- `assets/graphics/chain-rule-map.svg`
- `assets/graphics/derivative-toolkit-map.svg`
- `assets/graphics/derivative-sign-and-concavity.svg`
- `assets/graphics/optimization-workflow-map.svg`
- `assets/graphics/accumulation-function.svg`
- `assets/graphics/ftc-area-function.svg`
- `assets/graphics/integration-method-selector.svg`
- `assets/graphics/trapezoidal-rule.svg`
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- `assets/graphics/application-slicing-map.svg`
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- `assets/graphics/slope-field-logistic.svg`
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- `assets/graphics/directional-derivative-geometry.svg`
- `assets/graphics/area-foundations-diagram.svg`

- `assets/graphics/integration-slice-orders.svg`
- `assets/graphics/surface-normal-flux.svg`
- `assets/graphics/tetrahedron-box-volume.svg`
- `assets/graphics/vector-field-circulation.svg`
- `assets/graphics/vector-calculus-theorem-map.svg`
- `assets/graphics/parametric-polar-map.svg`
- `assets/graphics/curvature-osculating-circle.svg`
- `assets/graphics/second-order-oscillation-map.svg`
- `assets/graphics/damping-cases.svg`
- `assets/graphics/improper-integral-map.svg`
- `assets/graphics/single-variable-solution-flow.svg`
- `assets/graphics/advanced-calculus-solution-flow.svg`
- `assets/graphics/calculus-history-timeline.svg`
- `assets/graphics/exhaustion-to-integral.svg`
- `assets/graphics/local-to-global-theorem-family.svg`
- `assets/graphics/worked-solution-checklist.svg`
- `assets/graphics/review-cycle-and-retrieval.svg`
- `assets/graphics/calculus-history-branches.svg`
- `assets/graphics/application-translation-flow.svg`
- `assets/graphics/summary-checklist-map.svg`
- `assets/graphics/error-analysis-loop.svg`
- `assets/graphics/chapter-casebook-map.svg`
- `assets/graphics/exam-form-architecture.svg`
- `assets/graphics/practice-ladder-grid.svg`

Topical Index

This topical index groups major ideas by chapter and appendix so readers can relocate concepts quickly throughout the book.

A

B

C

D

E

F

G

H

I

J

L

M

N

O

P

Q

R

S

T

U

V

W