AP Calculus AB/BC

Compact Course Book

Limits, derivatives, integrals, differential equations, parametric and polar curves, and infinite series

What this book is

This is an **unofficial**, student-friendly AP Calculus companion. It is designed to do three jobs at once:

- teach the core ideas clearly,
- connect the topics to the current AP Calculus AB/BC structure, and
- give you enough practice and strategy to turn understanding into exam points.

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad \int_a^b f(x)\,dx \qquad \sum_{n=0}^{\infty} a_n$$

Prepared for independent study

Before you begin

This book is intentionally compact. It is not meant to replace your class notes, your teacher, or a full-length textbook. It is meant to help you move fast without getting sloppy.

How to use it

- If you are in **AP Calculus AB**, focus on Chapters 0–8, then read Chapters 11–12 and the appendices.
- If you are in **AP Calculus BC**, use the full book. BC contains all AB content plus extra integration techniques, arc length, parametric and polar work, vector-valued motion, and infinite sequences and series.
- Read with a pencil. For every worked example, cover the solution and do the setup yourself first.
- On AP-style work, always include notation, units when needed, and a sentence of interpretation when the problem is in context.

Roadmap

Track	Main chapters	Extra focus
AB	0–8, 11, 12, Appendices A–C	Limits, differentiation, integration, differential equations, applications, exam execution
BC	Everything in the book	BC-only integration techniques, arc length, parametric/polar/vector work, sequences and series

A note on AP alignment

The chapter sequence follows the current College Board AP Calculus course structure: eight units for AB and ten units for BC. Exam-format notes in Chapter 11 were checked against official College Board AP Central and AP Students pages current as of March 2026.

Preface

Calculus is the mathematics of change. That sounds abstract until you notice how often change is the real question: How fast is a quantity moving? When is something growing most quickly? How much has accumulated over time? When does a model stop behaving well? Calculus gives you a language for those questions.

The AP Calculus exams reward more than memorized formulas. They reward structure: reading a function carefully, choosing the right rule, writing the correct setup, interpreting units, and justifying conclusions. Students often know more mathematics than their scores show because they rush, skip notation, or fail to translate between graphs, tables, words, and formulas. This book is built to fix that.

Each chapter has four jobs:

1. explain the central idea in plain language,
2. show the algebra and notation that make the idea useful,
3. work a representative example in enough detail to imitate, and
4. give short practice so the topic sticks.

When you get stuck, do not immediately hunt for a trick. Ask the brutal but useful questions first: *What is changing? What is being asked? What representation do I have? What theorem or definition actually fits here?* Those four questions rescue a surprising number of AP problems.

Contents

Part I

Foundations

Chapter 0

Algebra and Function Toolkit

BRIDGE CHAPTER FOR AB AND BC

Learning goals

- Recognize the function families that appear constantly in AP Calculus.
- Move cleanly among formulas, graphs, tables, and verbal descriptions.
- Review algebra, trigonometry, exponentials, logarithms, and average rate of change.

0.1 The minimum algebra you need

Calculus is unforgiving about weak algebra. Most calculus mistakes are not really calculus mistakes at all; they are broken simplifications, bad inverse work, or forgotten function facts.

Here are the function families you should instantly recognize:

- **Polynomials**: smooth everywhere, no breaks, easy derivatives.
- **Rational functions**: ratios of polynomials, often with holes or vertical asymptotes.
- **Exponential functions**: constant percentage change, such as a^x or e^x.
- **Logarithmic functions**: inverses of exponentials, such as $\ln x$.
- **Trigonometric functions**: periodic behavior, especially $\sin x$ and $\cos x$.
- **Piecewise functions**: rules that change by interval; these are where one-sided limits matter.

Function facts to keep in your head

Composition	$(f \circ g)(x) = f(g(x))$
Inverse relation	$f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$
Average rate of change	$\frac{f(b) - f(a)}{b - a}$
Difference of squares	$a^2 - b^2 = (a - b)(a + b)$
Exponent rule	$a^{m+n} = a^m a^n$
Log rule	$\ln(ab) = \ln a + \ln b$

0.2 Domains, inverses, and transformations

When you read a calculus question, one of the fastest wins is identifying the domain and the shape of the graph before touching any formal calculation. Ask:

- What inputs are allowed?
- Is the function increasing or decreasing?
- Does it repeat, level off, or blow up?
- Is there symmetry?

For inverses, the central habit is simple: write $y = f(x)$, swap x and y, and solve for y. Then *check* by composition. Students skip the check and then carry a wrong inverse into an entire derivative problem.

Worked example: inverse and composition

Let $f(x) = 3x - 5$ and $g(x) = x^2 + 1$.
Then

$$(f \circ g)(x) = f(g(x)) = 3(x^2 + 1) - 5 = 3x^2 - 2.$$

To find f^{-1}, write $y = 3x - 5$, swap x and y, and solve:

$$x = 3y - 5 \qquad \Rightarrow \qquad y = \frac{x + 5}{3}.$$

So $f^{-1}(x) = \frac{x + 5}{3}$.

0.3 Trig, exponentials, and logs

The AP exams assume you already know the graphs and basic identities for $\sin x$, $\cos x$, e^x, and $\ln x$. At minimum, remember:

$$\sin^2 x + \cos^2 x = 1, \qquad e^{\ln x} = x \ (x > 0), \qquad \ln(e^x) = x.$$

Also remember the special angle values for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, and $\frac{\pi}{2}$. If your trig fundamentals are shaky, calculus will feel harder than it actually is.

Common traps before calculus even starts

- Treating $\dfrac{a+b}{c+d}$ as $\dfrac{a}{c} + \dfrac{b}{d}$.
- Forgetting that $\ln(x)$ is defined only for $x > 0$.
- Mixing up radians and degrees. AP Calculus uses **radians**.
- Solving for an inverse without checking whether the original function is one-to-one on the relevant domain.

0.4 Average rate of change: the bridge to derivatives

Before the derivative comes the average rate of change:

$$\frac{f(b) - f(a)}{b - a}.$$

This is the slope of the secant line through $(a, f(a))$ and $(b, f(b))$. Derivatives arise when that interval shrinks.

Worked example: average rate of change

For $f(x) = x^2 - 4x + 1$, the average rate of change on $[1, 4]$ is

$$\frac{f(4) - f(1)}{4 - 1} = \frac{(16 - 16 + 1) - (1 - 4 + 1)}{3} = \frac{1 - (-2)}{3} = 1.$$

Interpretation: on average, the output increases by 1 unit for each 1-unit increase in x over that interval.

Practice

1. Let $f(x) = 2x + 1$ and $g(x) = x^2 - 3$. Find $(f \circ g)(2)$ and $(g \circ f)(2)$.
2. Find the inverse of $f(x) = \dfrac{x-4}{3}$.
3. Simplify $\ln(e^{3x}) + \ln 5$.
4. Solve $\sin x = \dfrac{1}{2}$ on $[0, 2\pi]$.
5. Find the average rate of change of $f(x) = x^2 + 1$ on $[1, 3]$.
6. A population grows from 1200 to 1680 over 4 years. What is the average rate of change in population per year?

Chapter takeaways

- Function language matters: domain, inverse, composition, and transformation ideas show up all year.
- Average rate of change is the secant-line idea that becomes the derivative when the interval collapses.
- Logs, exponentials, and trig are not side topics in AP Calculus; they are everywhere.

Part II

AP Calculus AB Core

Chapter 1

Limits and Continuity

AP Unit 1 for AB and BC

Learning goals

- Interpret a limit from formulas, graphs, tables, and words.
- Compute limits using direct substitution, algebraic simplification, and one-sided reasoning.
- Recognize continuity, removable discontinuities, vertical asymptotes, and situations where the Intermediate Value Theorem applies.

1.1 What a limit actually says

The statement

$$\lim_{x \to a} f(x) = L$$

means that when x gets close to a, the function values get close to L. The key point is that a limit is about *nearby* behavior, not necessarily the value at the point itself.

Three things can happen at $x = a$ and the limit may still exist:

- $f(a)$ exists and equals L,
- $f(a)$ exists but is not L,
- $f(a)$ does not exist at all.

Core definitions

Limit	$\lim_{x\to a} f(x) = L$
Right-hand limit	$\lim_{x\to a^+} f(x)$
Left-hand limit	$\lim_{x\to a^-} f(x)$
Continuity at a	$\lim_{x\to a} f(x) = f(a)$

A two-sided limit exists only when the left-hand and right-hand limits both exist and are equal.

1.2 How to compute limits

Start with the obvious move: try direct substitution. If it works, you are done. If substitution gives an indeterminate form such as $\frac{0}{0}$, then simplify the expression with algebra.

Worked example: removing a hole

Evaluate

$$\lim_{x\to 2} \frac{x^2-4}{x-2}.$$

Direct substitution gives $\frac{0}{0}$, so factor:

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2 \qquad (x \neq 2).$$

Now take the limit of the simpler expression:

$$\lim_{x\to 2}(x+2) = 4.$$

The graph has a hole at $x = 2$, but the limit still exists and equals 4.

Other common algebra moves:

- factor polynomials,
- combine rational expressions,
- rationalize with a conjugate,
- use trig identities when appropriate.

A famous special limit is

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

You should know it and recognize when a problem can be rearranged into that form.

1.3 One-sided limits and infinite limits

Piecewise functions and absolute value functions often force you to examine the left side and the right side separately.

Worked example: one-sided limit

Let

$$f(x) = \frac{|x|}{x}.$$

For $x > 0$, $f(x) = 1$, and for $x < 0$, $f(x) = -1$. Therefore,

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1 \qquad \text{and} \qquad \lim_{x \to 0^+} \frac{|x|}{x} = 1.$$

Since these one-sided limits are different, the two-sided limit does not exist.

An *infinite limit* occurs when function values grow without bound:

$$\lim_{x \to a} f(x) = \infty \quad \text{or} \quad -\infty.$$

This usually signals a vertical asymptote at $x = a$.

1.4 Continuity and important theorems

A function is continuous at a if:

1. $f(a)$ exists,
2. $\lim_{x \to a} f(x)$ exists,
3. and the two are equal.

Typical discontinuities:

- **Removable**: a hole in the graph.
- **Jump**: left and right limits exist but are not equal.
- **Infinite**: vertical asymptote behavior.

Two theorems you should know

Intermediate Value Theorem (IVT). If f is continuous on $[a, b]$ and N is between $f(a)$ and $f(b)$, then there exists some c in $[a, b]$ with $f(c) = N$.

Squeeze Theorem. If $g(x) \le f(x) \le h(x)$ near a and

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L,$$

then $\lim_{x \to a} f(x) = L$.

Common traps

- Plugging in a value, getting $\frac{0}{0}$, and deciding the limit does not exist. It only means you need more work.
- Confusing "the limit does not exist" with "the function is undefined." Those are different statements.
- Using the IVT without checking continuity on the full interval.

Practice

1. Evaluate $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$.
2. Evaluate $\lim\limits_{x \to 0} \dfrac{\sqrt{x + 4} - 2}{x}$.
3. Determine whether $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ exists.
4. Find $\lim\limits_{x \to 2^-} \dfrac{1}{x - 2}$ and $\lim_{x \to 2^+} \frac{1}{x-2}$.
5. Determine whether the function

$$f(x) = \begin{cases} x + 1, & x < 2, \\ 5, & x = 2, \\ x^2 - 1, & x > 2 \end{cases}$$

is continuous at $x = 2$.

6. Use the Squeeze Theorem to evaluate $\lim\limits_{x \to 0} x^2 \sin\left(\dfrac{1}{x}\right)$.

7. A continuous function satisfies $f(1) = -2$ and $f(4) = 5$. What can you conclude from the IVT?

Chapter takeaways

- A limit is about nearby behavior, not necessarily the value at the point.
- Direct substitution is the first move, not the only move.
- Continuity is the gatekeeper for powerful theorems like the IVT.

Chapter 2

Differentiation: Definition and Fundamental Properties

AP Unit 2 for AB and BC

Learning goals

- Interpret the derivative as an instantaneous rate of change and as the slope of a tangent line.
- Use the limit definition of the derivative.
- Differentiate common functions using basic derivative rules.

2.1 Why derivatives matter

An average rate of change looks at an interval. A derivative looks at a *moment*. If

$$f'(a)$$

exists, it tells you the instantaneous rate of change of f with respect to x at $x = a$, and it also gives the slope of the tangent line to the graph at that point.

Two equivalent derivative definitions

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

and, equivalently,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Both definitions say the same thing: the derivative is the limit of secant slopes.

2.2 Derivative from first principles

You do not want to compute every derivative from the limit definition forever. But you *do* want to understand where the rules come from.

Worked example: derivative by definition

Let $f(x) = x^2$. Then

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}.$$

Simplify:

$$f'(x) = \lim_{h\to 0}(2x + h) = 2x.$$

That is the slope at every point on the parabola.

2.3 Basic derivative rules

Once the limit idea is clear, use rules aggressively.

Core derivative formulas

Constant	$\frac{d}{dx}[c] = 0$
Power rule	$\frac{d}{dx}[x^n] = nx^{n-1}$
Constant multiple	$\frac{d}{dx}[cf(x)] = cf'(x)$
Sum / difference	$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
Exponential	$\frac{d}{dx}[e^x] = e^x$
Logarithm	$\frac{d}{dx}[\ln x] = \frac{1}{x}$
Trig	$\frac{d}{dx}[\sin x] = \cos x, \quad \frac{d}{dx}[\cos x] = -\sin x$

Two more rules matter immediately:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, \qquad g(x) \neq 0.$$

Worked example: multiple rules at once

Differentiate

$$y = (x^2 + 1)\sin x.$$

This is a product, so use the product rule:

$$y' = (2x)\sin x + (x^2 + 1)\cos x.$$

Do not expand first unless expansion makes the derivative easier.

2.4 Tangents, motion, and notation

If $y = f(x)$, then the tangent line at $x = a$ is

$$y - f(a) = f'(a)(x - a).$$

The same derivative can appear in many notations:

$$f'(x), \qquad y', \qquad \frac{dy}{dx}, \qquad \frac{d}{dx}[f(x)].$$

In motion problems, if position is $s(t)$, then

$$v(t) = s'(t) \qquad \text{and} \qquad a(t) = v'(t) = s''(t).$$

Common traps

- Using the product rule where the power rule would work, or vice versa.
- Forgetting that the derivative of $\cos x$ is $-\sin x$.
- Writing a tangent-line slope correctly but forgetting to write the line itself.
- Dropping units in rate problems.

Practice

1. Use the limit definition to find the derivative of $f(x) = 3x + 1$.
2. Differentiate $f(x) = 5x^4 - 3x^2 + 7$.
3. Differentiate $y = x^3 e^x$.
4. Differentiate $y = \dfrac{x^2 + 1}{x - 1}$.

5. Find the equation of the tangent line to $y = x^2 - 1$ at $x = 2$.
6. If $s(t) = t^3 - 6t^2 + 9t$, find $v(t)$ and $a(t)$.
7. Differentiate $y = \sin x + \ln x$.

Chapter takeaways

- The derivative is a limit of slopes; the rules are shortcuts built from that idea.
- Always identify structure first: sum, product, quotient, or composition.
- In applications, derivatives carry units and meaning; do not treat them as naked symbols.

Chapter 3

Differentiation: Composite, Implicit, and Inverse Functions

AP Unit 3 for AB and BC

Learning goals

- Use the chain rule correctly and without hesitation.
- Differentiate implicitly defined relations.
- Find derivatives of inverse functions, including inverse trig functions, and use logarithmic differentiation when it simplifies the work.

3.1 The chain rule

Whenever one function sits inside another, you are looking at a composition. That means the chain rule is probably coming.

If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

The fastest way to stay organized is to say out loud what the *outside* function is and what the *inside* function is.

Worked example: chain rule

Differentiate

$$y = (3x^2 - 1)^5.$$

The outside function is "raise to the fifth power," and the inside function is $3x^2 - 1$. So

$$y' = 5(3x^2 - 1)^4 \cdot (6x) = 30x(3x^2 - 1)^4.$$

3.2 Implicit differentiation

Not every relation comes solved for y. When x and y are tangled together, differentiate both sides with respect to x and remember that y is a function of x.

Worked example: implicit relation

For the circle

$$x^2 + y^2 = 25,$$

differentiate both sides:

$$2x + 2y\frac{dy}{dx} = 0.$$

Now solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{x}{y}.$$

That extra factor of $\frac{dy}{dx}$ appears whenever you differentiate a term containing y.

3.3 Inverse functions

If f has an inverse and $f(a) = b$, then

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

Equivalent form:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Worked example: derivative of an inverse at a point

Suppose $f(2) = 5$ and $f'(2) = 7$. Then

$$(f^{-1})'(5) = \frac{1}{f'(2)} = \frac{1}{7}.$$

You do *not* plug 5 into f'. You use the matching input-output pair.

You should also know the inverse trig derivatives:

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}.$$

Those formulas appear often enough that they are worth memorizing.

3.4 Logarithmic differentiation

Logarithmic differentiation is the smart move when powers and products are messy.

Worked example: x^x

Let

$$y = x^x \qquad (x > 0).$$

Take natural logs:

$$\ln y = x \ln x.$$

Differentiate:

$$\frac{1}{y}\frac{dy}{dx} = \ln x + 1.$$

Multiply by $y = x^x$:

$$\frac{dy}{dx} = x^x(\ln x + 1).$$

Common traps

- Forgetting the inside derivative when using the chain rule.
- In implicit differentiation, differentiating y^2 as $2y$ instead of $2y\,\frac{dy}{dx}$.
- Mixing up the input and output in inverse-function derivative questions.
- Using logarithmic differentiation and then forgetting to solve back for $\frac{dy}{dx}$.

Practice

1. Differentiate $y = (x^2+4)^6$.
2. Differentiate $y = \sqrt{1+3x^2}$.

3. For the relation $x^2 + xy + y^2 = 7$, find $\dfrac{dy}{dx}$.

4. Suppose $f(3) = 2$ and $f'(3) = 5$. Find $(f^{-1})'(2)$.

5. Differentiate $y = \arcsin(2x)$.

6. Use logarithmic differentiation to find the derivative of $y = x^2(x-1)^3$.

7. Differentiate $y = x^x$.

Chapter takeaways

- Composition means chain rule.
- Implicit differentiation treats y as a function of x, so every derivative of a y-term picks up a factor of $\dfrac{dy}{dx}$.
- Inverse-function derivatives are reciprocal slopes at matching points, not at matching coordinates.

Chapter 4

Contextual Applications of Differentiation

AP Unit 4 for AB and BC

Learning goals

- Interpret derivatives in context, including units and meaning.
- Solve motion and related-rates problems.
- Use local linearity, linearization, and differentials for approximation.

4.1 Derivatives in context

In a pure function problem, a derivative is a slope. In a context problem, a derivative is a rate with units.

If $V(t)$ is water volume in liters, then $V'(t)$ has units of liters per minute. If $P(x)$ is profit in dollars, then $P'(x)$ has units of dollars per item. AP readers want both the number and the meaning.

Interpretation template

If $f'(a) = k$, then at the instant or input $x = a$, the quantity f is changing at a rate of k *units of output per unit of input*.

Tables matter too. If you are given values rather than a formula, a derivative is often estimated with a difference quotient. The closer the surrounding points are to the target, the better the estimate.

4.2 Motion along a line

If position is $s(t)$, then

$$v(t) = s'(t) \qquad \text{and} \qquad a(t) = v'(t) = s''(t).$$

Three useful interpretations:

- $v(t) > 0$: moving to the right or increasing position.
- $v(t) < 0$: moving to the left or decreasing position.
- Speed is $|v(t)|$, not $v(t)$.

Worked example: motion

A particle has position

$$s(t) = t^3 - 6t^2 + 9t.$$

Then

$$v(t) = 3t^2 - 12t + 9 \qquad \text{and} \qquad a(t) = 6t - 12.$$

At $t = 1$, the particle has velocity $v(1) = 0$ and acceleration $a(1) = -6$. The particle is momentarily at rest there, but the acceleration is still negative.

4.3 Related rates

Related-rates problems look scary when students rush. The structure is simple:

1. Draw the situation and label variables.
2. Write an equation connecting those variables.
3. Differentiate with respect to time.
4. Substitute the known values only after differentiating.

Worked example: expanding sphere

A sphere expands so that its radius increases at 2 cm/s. How fast is the volume changing when $r = 3$ cm?
Use

$$V = \frac{4}{3}\pi r^3.$$

Differentiate with respect to time:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Now substitute $r = 3$ and $\frac{dr}{dt} = 2$:

$$\frac{dV}{dt} = 4\pi(3)^2(2) = 72\pi \text{ cm}^3/\text{s}.$$

4.4 Linearization and differentials

Near a point $x = a$, differentiable functions behave almost like lines. The tangent line gives a local approximation:

$$L(x) = f(a) + f'(a)(x - a).$$

This is called the *linearization* of f at $x = a$.

If dx is a small change in x, then the differential

$$dy = f'(x)\,dx$$

approximates the actual change Δy.

Worked example: linearization

Use linearization to approximate $\sqrt{4.1}$.
Let $f(x) = \sqrt{x}$ and linearize at $a = 4$. Then

$$f(4) = 2, \qquad f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(4) = \frac{1}{4}.$$

So

$$L(x) = 2 + \frac{1}{4}(x - 4).$$

At $x = 4.1$,

$$L(4.1) = 2 + \frac{1}{4}(0.1) = 2.025.$$

Therefore $\sqrt{4.1} \approx 2.025$.

Common traps

- Writing the units for the original quantity instead of the derivative's units.
- Substituting numbers into a related-rates problem before differentiating.
- Confusing velocity with speed.
- Using linearization far away from the base point.

Practice

1. A tank contains $V(t)$ liters of water. What are the units of $V'(t)$ if t is measured in minutes?
2. A particle has position $s(t) = t^2 - 8t + 5$. Find its velocity and acceleration.
3. Estimate $f'(3)$ from the table if $f(2.9) = 4.1$, $f(3.0) = 4.4$, and $f(3.1) = 4.7$.
4. A circular ripple has radius increasing at 3 cm/s. How fast is the area changing when $r = 5$ cm?
5. Use linearization at $x = 9$ to approximate $\sqrt{9.2}$.
6. If the side length x of a cube changes by $dx = 0.02$ cm when $x = 4$ cm, use differentials to approximate the change in volume.

Chapter takeaways

- In context, a derivative is a rate with units and interpretation.
- Related rates are mostly organization problems, not genius problems.
- Linearization is a tangent-line approximation; it is best near the point of tangency.

Chapter 5

Analytical Applications of Differentiation

AP Unit 5 for AB and BC

Learning goals

- Use derivatives to locate extrema, intervals of increase and decrease, and concavity.
- Apply the Extreme Value Theorem, Rolle's Theorem, and the Mean Value Theorem.
- Solve optimization problems and justify conclusions.

5.1 Critical points and extrema

A *critical point* occurs where $f'(x) = 0$ or where $f'(x)$ does not exist, provided x is in the domain of f.

Critical points are candidates for local extrema, but they are not automatically maxima or minima. You still need a test.

The three essential tests

First derivative test. If f' changes from positive to negative, f has a local maximum. If f' changes from negative to positive, f has a local minimum.

Second derivative test. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c. If $f''(c) < 0$, then f has a local maximum at c.

Closed-interval method. To find an absolute max or min on $[a, b]$, check all critical points in the interval and the endpoints.

5.2 Theorems that justify your reasoning

Extreme Value Theorem (EVT). If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum and an absolute minimum on that interval.

Rolle's Theorem. If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there exists some c in (a, b) with $f'(c) = 0$.

Mean Value Theorem (MVT). If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists some c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The exam often wants two things: check the theorem's conditions, then state the conclusion in a complete sentence.

5.3 Increasing, decreasing, and concavity

The sign of f' controls whether a function rises or falls:

$$f'(x) > 0 \Rightarrow \text{increasing}, \qquad f'(x) < 0 \Rightarrow \text{decreasing}.$$

The sign of f'' controls concavity:

$$f''(x) > 0 \Rightarrow \text{concave up}, \qquad f''(x) < 0 \Rightarrow \text{concave down}.$$

An *inflection point* is where concavity changes.

Worked example: classify a critical point

Let

$$f(x) = x^3 - 3x.$$

Then

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1),$$

so the critical points are $x = \pm 1$.
Next,

$$f''(x) = 6x.$$

At $x = -1$, $f''(-1) = -6 < 0$, so f has a local maximum there. At $x = 1$, $f''(1) = 6 > 0$, so f has a local minimum there.

5.4 Optimization

Optimization problems are just applied extrema problems. The structure:

1. Draw a picture if one exists.
2. Write the quantity to be optimized.
3. Use the constraints to rewrite it in one variable.
4. Differentiate and solve for critical points.
5. Check that the answer makes sense in the original context.

Worked example: fixed perimeter rectangle

A rectangle has perimeter 40. What dimensions maximize the area?
Let the sides be x and y. Then

$$2x + 2y = 40 \Rightarrow y = 20 - x.$$

Area:

$$A(x) = x(20 - x) = 20x - x^2.$$

Differentiate:

$$A'(x) = 20 - 2x.$$

Set equal to zero:

$$20 - 2x = 0 \Rightarrow x = 10.$$

Then $y = 10$. The area is maximized by a square.

Common traps

- Stopping after finding critical points without classifying them.
- Using the second derivative test when $f''(c) = 0$ and pretending that proves something.
- Forgetting to check endpoints in closed-interval absolute-extrema problems.
- In optimization, optimizing the wrong quantity.

Practice

1. For $f(x) = x^3 - 12x$, find all critical points.
2. Determine the intervals on which $f(x) = x^3 - 3x^2$ is increasing and decreasing.
3. Find the local extrema of $f(x) = x^4 - 4x^2$.
4. State whether the hypotheses of the Mean Value Theorem are satisfied for $f(x) = x^2$ on $[1, 5]$, and find the corresponding value(s) of c.
5. Find the intervals of concavity for $f(x) = x^3 - 6x^2 + 9x$.
6. A rectangle has area 48 square units. What dimensions minimize its perimeter?
7. Find the absolute maximum and minimum of $f(x) = x^2 - 4x + 1$ on $[-1, 4]$.

Chapter takeaways

- Derivatives do not just compute slopes; they reveal a function's behavior.
- Theorems like EVT and MVT matter because AP readers want justified conclusions, not guesses.
- Optimization is calculus plus restraint: one variable, correct objective, correct conclusion.

Chapter 6

Integration and Accumulation of Change

AP Unit 6 for AB and BC; includes BC-only integration techniques

Learning goals

- Interpret a definite integral as accumulation and net change.
- Use antiderivatives and the Fundamental Theorem of Calculus.
- Approximate integrals numerically and use substitution.
- For BC: use integration by parts, partial fractions, improper integrals, and technique selection.

6.1 Antiderivatives and indefinite integrals

An antiderivative of f is a function F such that $F'(x) = f(x)$. The indefinite integral notation

$$\int f(x)\,dx$$

represents the family of all antiderivatives of f:

$$\int f(x)\,dx = F(x) + C.$$

Essential antiderivatives

Power rule	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
Reciprocal	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
Exponential	$\int e^x\,dx = e^x + C$
Trig	$\int \sin x\,dx = -\cos x + C, \quad \int \cos x\,dx = \sin x + C$

The constant of integration matters. On AP free response, leaving it off in an indefinite integral can cost a point.

6.2 Definite integrals and Riemann sums

The definite integral

$$\int_a^b f(x)\,dx$$

gives the net signed area, or more generally the accumulated change of the rate f from $x = a$ to $x = b$.

Riemann sums approximate that accumulation:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Left, right, and midpoint sums all approximate the same quantity from different sample points.

Worked example: net change

If a particle has velocity $v(t) = 2t + 1$ for $0 \le t \le 3$, then the change in position is

$$\int_0^3 (2t+1)\,dt = [t^2 + t]_0^3 = (9+3) - 0 = 12.$$

The particle's position changes by 12 units over that interval.

6.3 The Fundamental Theorem of Calculus

The FTC has two parts you should connect, not memorize separately.

Fundamental Theorem of Calculus

Part 1. If

$$g(x) = \int_a^x f(t)\,dt,$$

then

$$g'(x) = f(x)$$

provided f is continuous.

Part 2. If $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Worked example: accumulation function

Let

$$g(x) = \int_1^x \sqrt{t^2+4}\,dt.$$

By FTC Part 1,

$$g'(x) = \sqrt{x^2+4}.$$

If you need $g'(3)$, you do *not* integrate. You simply evaluate the integrand at $x = 3$:

$$g'(3) = \sqrt{13}.$$

6.4 Substitution and numerical integration

Substitution reverses the chain rule. If an integrand looks like "inside function" times "its derivative," try u-substitution.

Worked example: substitution

Evaluate

$$\int 2x\cos(x^2)\,dx.$$

Let $u = x^2$, so $du = 2x\,dx$. Then

$$\int 2x\cos(x^2)\,dx = \int \cos u\,du = \sin u + C = \sin(x^2) + C.$$

For numerical approximation, the trapezoidal rule on equal-width subintervals is

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

If a function is concave up, the trapezoidal rule tends to overestimate; if concave down, it tends to underestimate.

6.5 BC-only extension: choosing techniques

BC asks you to choose among several antidifferentiation tools. The point is not to show off techniques. The point is to match the integrand's structure.

BC-only technique guide

- **Substitution**: composition structure, especially "inside" times "derivative of inside."
- **Integration by parts**: products such as xe^x, $x \sin x$, or $\ln x$.
- **Linear partial fractions**: rational functions decomposed into simpler fractions.
- **Improper integrals**: infinite interval or unbounded integrand; rewrite as a limit.

Worked example: integration by parts

Evaluate

$$\int xe^x \, dx.$$

Let $u = x$ and $dv = e^x \, dx$. Then $du = dx$ and $v = e^x$. Integration by parts gives

$$\int xe^x \, dx = xe^x - \int e^x \, dx = xe^x - e^x + C = e^x(x-1) + C.$$

Worked example: partial fractions

Evaluate

$$\int \frac{1}{x(x+1)} \, dx.$$

Decompose:

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}.$$

Then

$$\int \frac{1}{x(x+1)}\,dx = \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx = \ln|x| - \ln|x+1| + C.$$

An improper integral, such as

$$\int_1^\infty \frac{1}{x^2}\,dx,$$

must be written as a limit:

$$\lim_{b\to\infty} \int_1^b x^{-2}\,dx = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty} \left(1 - \frac{1}{b}\right) = 1.$$

Because the limit is finite, the integral converges.

Common traps

- Forgetting the constant of integration on an indefinite integral.
- Integrating when FTC Part 1 only asks for the derivative of an accumulation function.
- Doing substitution without changing the differential.
- For BC, using integration by parts or partial fractions when a simple substitution would have worked faster.
- Treating an improper integral like an ordinary one without writing a limit.

Practice

1. Find $\int (6x^2 - 4x + 1)\,dx$.
2. Evaluate $\int_0^2 (3x - 1)\,dx$.
3. If $g(x) = \int_2^x \cos(t^2)\,dt$, find $g'(x)$ and $g'(2)$.
4. Evaluate $\int 4x(x^2 + 1)^3\,dx$.
5. Use the trapezoidal rule with $n = 2$ to approximate $\int_0^2 x^2\,dx$.

BC Evaluate $\int x \cos x\,dx$.

BC Evaluate $\int \frac{3}{x(x+3)}\,dx$.

BC Determine whether $\int_1^\infty \frac{1}{x}\,dx$ converges or diverges.

Chapter takeaways

- Definite integrals model accumulation, and the FTC is the bridge between accumulation and antiderivatives.
- Substitution reverses the chain rule.
- BC students must learn to select the right antidifferentiation tool, not just execute one memorized method.

Chapter 7

Differential Equations

AP Unit 7 for AB and BC

Learning goals

- Read differential equations as statements about rates of change.
- Solve separable differential equations and use initial conditions.
- Interpret slope fields, exponential/logistic models, and Euler's method.

7.1 What a differential equation means

A differential equation tells you how a quantity changes, not necessarily the quantity itself. For example,

$$\frac{dy}{dx} = 3y$$

says the rate of change of y is three times its current value. Exponential growth is hiding in that statement.

Sometimes you are asked for the family of solutions; sometimes you are asked for the particular solution satisfying an initial condition such as $y(0) = 5$.

7.2 Slope fields

A slope field is a picture of many tiny tangent segments. You do not need a formula for y to read behavior from it.

When interpreting a slope field, ask:

- Where are the slopes zero?

- Where are they positive or negative?
- Do solution curves level off, grow quickly, or move toward an equilibrium?

You can often tell whether a solution passing through a given point will increase, decrease, or stay nearly flat without solving the differential equation exactly.

7.3 Separable equations

If you can rewrite the equation so all y terms are with dy and all x terms are with dx, separate and integrate.

Worked example: separable equation

Solve

$$\frac{dy}{dx} = 2xy \qquad \text{with} \qquad y(0) = 3.$$

Separate:

$$\frac{1}{y}\, dy = 2x\, dx.$$

Integrate:

$$\int \frac{1}{y}\, dy = \int 2x\, dx \quad \Rightarrow \quad \ln|y| = x^2 + C.$$

Exponentiate:

$$y = Ce^{x^2}.$$

Use $y(0) = 3$ to get $C = 3$, so

$$y = 3e^{x^2}.$$

7.4 Exponential and logistic models

Two model families appear often.

Exponential model:

$$\frac{dy}{dt} = ky \quad \Rightarrow \quad y(t) = Ce^{kt}.$$

If $k > 0$, the quantity grows. If $k < 0$, it decays.

Logistic model:

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right).$$

Here L is the carrying capacity. When $0 < y < L$, the quantity grows; near L, the growth slows.

You do not always need the explicit solution of a logistic differential equation. Often the AP question only wants equilibrium values, increasing/decreasing behavior, or a numerical estimate.

7.5 Euler's method

Euler's method approximates a solution using tangent-line steps:

$$y_{n+1} = y_n + f(x_n, y_n)\Delta x.$$

It is basically repeated linearization.

Worked example: one Euler step

Suppose

$$\frac{dy}{dx} = x + y, \qquad y(0) = 1.$$

Use Euler's method with step size $\Delta x = 0.2$ to estimate $y(0.2)$.
At $(x_0, y_0) = (0, 1)$, the slope is

$$f(0, 1) = 0 + 1 = 1.$$

So

$$y_1 = y_0 + f(x_0, y_0)\Delta x = 1 + 1(0.2) = 1.2.$$

Thus $y(0.2) \approx 1.2$.

Common traps

- Separating variables incorrectly.
- Forgetting absolute values in $\int \frac{1}{y}\, dy = \ln|y| + C$.
- Solving a differential equation correctly but ignoring the initial condition.
- In Euler's method, using the wrong slope or forgetting to update the point after each step.

Practice

1. Explain in words what $\frac{dP}{dt} = 0.3P$ says about the population P.
2. Solve $\frac{dy}{dx} = 4x$ with $y(1) = 2$.

3. Solve $\frac{dy}{dx} = y$ with $y(0) = 5$.
4. For the logistic equation $\frac{dP}{dt} = 0.2P\left(1 - \frac{P}{100}\right)$, what are the equilibrium solutions?
5. Use Euler's method with step size 0.1 for one step to estimate $y(0.1)$ if $\frac{dy}{dx} = x - y$ and $y(0) = 2$.
6. If a solution curve to $\frac{dy}{dx} = y - 4$ passes through a point with $y = 1$, is the solution increasing or decreasing there?

Chapter takeaways

- Differential equations describe change rules; solutions are functions that obey those rules.
- Separation of variables and initial conditions are the core symbolic tools in this unit.
- Euler's method is just repeated tangent-line approximation.

Chapter 8

Applications of Integration

AP Unit 8 for AB and BC; includes BC-only arc length

Learning goals

- Use definite integrals to compute net change, area, volume, and average value.
- Set up area-between-curves and volume formulas from geometry.
- For BC: compute arc length and distance traveled from integral expressions.

8.1 Net change and accumulated quantities

If $R(t)$ is a rate, then

$$\int_a^b R(t)\,dt$$

gives the total change in the underlying quantity from $t = a$ to $t = b$. This is the *Net Change Theorem*. Many word problems are only disguised versions of that idea.

If $v(t)$ is velocity, then

$$\int_a^b v(t)\,dt$$

is displacement, while

$$\int_a^b |v(t)|\,dt$$

is total distance traveled.

8.2 Area between curves

For curves $y = f(x)$ and $y = g(x)$ on an interval where $f(x) \geq g(x)$,

$$\text{Area} = \int_a^b [f(x) - g(x)]\,dx.$$

The phrase to remember is **top minus bottom**. If you integrate with respect to y, it becomes **right minus left**.

Worked example: area between curves

Find the area enclosed by

$$y = x \qquad \text{and} \qquad y = x^2.$$

First find intersections:

$$x = x^2 \Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1.$$

On $[0, 1]$, the line $y = x$ lies above $y = x^2$. So

$$A = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

8.3 Volumes

The two standard AP volume methods are disks/washers and known cross sections.

Disk/washer method. If cross sections perpendicular to the x-axis are circles,

$$V = \pi \int_a^b [R(x)]^2\,dx$$

for disks, or

$$V = \pi \int_a^b \left(R(x)^2 - r(x)^2\right) dx$$

for washers.

Known cross sections. If the cross-sectional area is $A(x)$, then

$$V = \int_a^b A(x)\,dx.$$

Worked example: volume by disks

Find the volume when the region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$ is revolved about the x-axis.
A cross section perpendicular to the x-axis is a disk of radius $\sqrt{x}$, so

$$V = \pi \int_0^4 (\sqrt{x})^2 \, dx = \pi \int_0^4 x \, dx = \pi \left[\frac{x^2}{2} \right]_0^4 = 8\pi.$$

8.4 Average value of a function

The average value of f on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) \, dx.$$

This formula matters both computationally and conceptually. It tells you the constant height whose rectangular area matches the accumulated area of the function over the interval.

8.5 BC-only extension: arc length

BC adds one more major geometry application. If a smooth planar curve is given by $y = f(x)$ on $[a, b]$, then its arc length is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx.$$

If position along a line is given by $x(t)$, total distance traveled from $t = a$ to $t = b$ is

$$\int_a^b |v(t)| \, dt.$$

These formulas look different, but both accumulate small pieces of length.

Worked example: arc length

Find the arc length of

$$y = \frac{x^2}{2}$$

from $x = 0$ to $x = 1$.

First compute

$$y' = x.$$

So

$$L = \int_0^1 \sqrt{1 + x^2}\, dx.$$

That exact antiderivative is not an AB-level skill, but the setup is the main point: the arc length is represented by the integral above. On BC, a calculator or other techniques may be used depending on the question.

Common traps

- Using "top minus bottom" on an interval where the curves switch order without splitting the integral.
- Forgetting to square the radius in disk/washer formulas.
- Confusing displacement with total distance traveled.
- For BC, writing the arc-length setup without the square root.

Practice

1. A particle has velocity $v(t) = t - 2$ on $[0, 5]$. Write an integral for the displacement and another for total distance traveled.
2. Find the area between $y = 2x$ and $y = x^2$ on the interval where they intersect.
3. Find the average value of $f(x) = x^2$ on $[0, 2]$.
4. Find the volume obtained by revolving the region under $y = x$ from 0 to 2 about the x-axis.
5. A solid has square cross sections perpendicular to the x-axis with side length x for $0 \leq x \leq 3$. Find its volume.

BC Write, but do not evaluate, an integral for the arc length of $y = \ln x$ from $x = 1$ to $x = 3$.

Chapter takeaways

- Applications of integration are geometry and accumulation problems in disguise.

- Area between curves is top minus bottom or right minus left, with interval checks built in.
- BC students should recognize arc length as another accumulation-of-small-pieces idea.

Part III

AP Calculus BC Extensions

Chapter 9

Parametric Equations, Polar Coordinates, and Vector-Valued Functions

AP Unit 9 for BC

Learning goals

- Differentiate and analyze parametrically defined curves.
- Interpret vector-valued position, velocity, speed, and acceleration.
- Work with polar curves, including slope and area.

9.1 Parametric curves

In parametric form, both coordinates depend on a parameter, usually t:

$$x = x(t), \qquad y = y(t).$$

This is ideal for motion, because time is built in naturally.

If $\dfrac{dx}{dt} \neq 0$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

That formula is not optional; it is the central tool of the chapter.

Worked example: slope of a parametric curve

Suppose

$$x = t^2 + 1, \qquad y = t^3 - 2.$$

Then

$$\frac{dx}{dt} = 2t, \qquad \frac{dy}{dt} = 3t^2,$$

so

$$\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2} \qquad (t \neq 0).$$

To get the second derivative,

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \bigg/ \frac{dx}{dt}.$$

That tells you concavity for a parametric curve.

9.2 Motion in the plane and vector-valued functions

A vector-valued position function can be written

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle.$$

Then

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$$

and

$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle x''(t), y''(t) \rangle.$$

Speed is the magnitude of velocity:

$$|\mathbf{v}(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}.$$

Worked example: speed

If

$$\mathbf{r}(t) = \langle t^2,\ t^3 \rangle,$$

then

$$\mathbf{v}(t) = \langle 2t,\ 3t^2 \rangle, \qquad |\mathbf{v}(t)| = \sqrt{4t^2 + 9t^4} = |t|\sqrt{4 + 9t^2}.$$

Integrating a vector-valued function is done component by component:

$$\int \langle f(t), g(t) \rangle \, dt = \left\langle \int f(t)\, dt, \int g(t)\, dt \right\rangle + \mathbf{C}.$$

9.3 Arc length for parametric curves

If a parametric curve is given by $x = x(t)$ and $y = y(t)$ on $[a, b]$, then

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

This is the natural extension of the arc-length formula from Unit 8.

9.4 Polar coordinates

A polar curve is written as $r = f(\theta)$. To convert between forms:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

The slope formula is

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

The area enclosed by a polar curve from $\theta = \alpha$ to $\theta = \beta$ is

$$A = \frac{1}{2}\int_\alpha^\beta [r(\theta)]^2\, d\theta.$$

Worked example: area in polar form

For the circle-like curve $r = 2$, the area from $\theta = 0$ to $\theta = \pi$ is

$$A = \frac{1}{2}\int_0^\pi 4\, d\theta = 2\pi.$$

That is exactly the area of a semicircle of radius 2.

Common traps

- Treating $\dfrac{dy}{dx}$ as $\dfrac{dy}{dt}$ in parametric problems.
- Forgetting that speed is the magnitude of velocity, not one component.
- Using the rectangular area formula instead of the polar area formula.
- In polar problems, failing to choose the correct interval of θ for the region.

Practice

1. For $x = t^2$ and $y = t + 1$, find $\dfrac{dy}{dx}$.
2. For $x = t^2$ and $y = t^3$, find $\dfrac{d^2y}{dx^2}$.
3. If $\mathbf{r}(t) = \langle 3t,\ t^2 \rangle$, find $\mathbf{v}(t)$ and the speed.
4. Write an integral for the arc length of $x = t$, $y = t^2$ on $0 \le t \le 1$.
5. For the polar curve $r = 1 + \cos\theta$, write an integral for the area enclosed over $0 \le \theta \le 2\pi$.
6. If $r = 2\sin\theta$, convert to rectangular form.

Chapter takeaways

- Parametric and vector-valued problems are motion problems in disguise more often than students realize.
- The formula $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ is the chapter's workhorse.
- Polar area is one-half the integral of r^2 over the correct angle interval.

Chapter 10

Infinite Sequences and Series

AP Unit 10 for BC

Learning goals

- Determine whether sequences and series converge or diverge.
- Use geometric, telescoping, comparison, alternating, ratio, and related tests.
- Represent functions with power series and Taylor/Maclaurin series.

10.1 Sequences versus series

A **sequence** is a list of terms:

$$a_1, a_2, a_3, \ldots$$

A **series** is a sum:

$$\sum_{n=1}^{\infty} a_n.$$

For a sequence, the central question is whether a_n approaches a finite limit as $n \to \infty$. For a series, the central question is whether the sequence of partial sums

$$S_N = \sum_{n=1}^{N} a_n$$

approaches a finite limit.

First fact to check

If $\sum a_n$ converges, then $a_n \to 0$.

$$\lim_{n\to\infty} a_n \neq 0 \quad \Rightarrow \quad \sum a_n \text{ diverges.}$$

This is the divergence test.

10.2 The series you should recognize immediately

Geometric series:

$$\sum_{n=0}^{\infty} ar^n$$

converges to

$$\frac{a}{1-r}$$

when $|r| < 1$, and diverges when $|r| \geq 1$.

p-series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.

Telescoping series: many terms cancel after expansion; write the first few partial sums to see the pattern.

10.3 Tests for convergence

When a series is not obviously geometric or telescoping, choose a test.

BC convergence toolkit

- **Comparison test**: compare to a known positive series.
- **Limit comparison test**: compare long-run behavior.
- **Integral test**: useful when terms come from a positive, decreasing, continuous function.
- **Alternating series test**: for terms of alternating sign that decrease to 0.
- **Ratio test**: especially useful with factorials and exponentials.

Worked example: geometric series

Determine whether

$$\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$$

converges, and if so find the sum.
This is geometric with $a = 1$ and $r = \frac{2}{3}$. Since $|r| < 1$, it converges and

$$\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{1-\frac{2}{3}} = 3.$$

Worked example: alternating series

Determine whether

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

converges.
The terms alternate, their magnitudes $\frac{1}{n}$ decrease, and $\frac{1}{n} \to 0$. Therefore the series converges by the Alternating Series Test. It does *not* converge absolutely, because $\sum \frac{1}{n}$ diverges.

10.4 Power series

A power series centered at a has the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n.$$

It converges on some interval around a, often described by a **radius of convergence** R and an interval of convergence found by testing endpoints separately.

Within its interval of convergence, a power series behaves beautifully: you can differentiate and integrate it term by term.

10.5 Taylor and Maclaurin series

A Taylor polynomial approximates a function near a center $x = a$:

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

A Maclaurin series is just a Taylor series centered at $a = 0$.

You should know these standard expansions:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Worked example: Maclaurin polynomial

Find the third-degree Maclaurin polynomial for e^x.
Because every derivative of e^x is e^x, each derivative at 0 equals 1. Therefore

$$T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

For alternating series and certain Taylor approximations, you can bound the error by the first omitted term. That makes approximation questions manageable rather than mystical.

Common traps

- Forgetting that $a_n \to 0$ is necessary but not sufficient for convergence.
- Using a convergence test that does not fit the series.
- Finding the radius of convergence correctly but forgetting to test endpoints.
- Writing a Taylor polynomial without the factorials.

Practice

1. Determine whether the sequence $a_n = \dfrac{3n+1}{n+4}$ converges, and if so find its limit.
2. Determine whether $\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n$ converges, and if so find its sum.
3. Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge?
4. Determine whether $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges or diverges.

5. Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely, conditionally, or diverges.

6. Use the ratio test to determine whether $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges for all real x.

7. Find the second-degree Maclaurin polynomial for $\cos x$.

8. Write the first four nonzero terms of the Maclaurin series for $\sin x$.

Chapter takeaways

- Sequences ask whether terms settle down; series ask whether partial sums settle down.
- Recognize geometric and p-series instantly, then choose the lightest correct test for everything else.
- Taylor and Maclaurin series let you replace hard functions with polynomials near a chosen center.

Part IV

Exam Playbook

Chapter 11

Exam Structure, Scoring Habits, and Strategy

CURRENT EXAM-FORMAT NOTES CHECKED AGAINST COLLEGE BOARD PAGES IN MARCH 2026

Learning goals

- Know the current AP Calculus exam format and how time is divided.
- Understand the biggest differences between AB and BC.
- Write solutions the way AP readers award points.

11.1 AB versus BC at a glance

AP Calculus BC contains all AB content plus BC-only topics. In practical terms:

- **AB** covers limits, derivatives, integrals, differential equations, and major applications.
- **BC** adds extra integration techniques, arc length, parametric and polar work, vector-valued functions, and infinite sequences and series.
- BC students receive a **Calculus AB subscore** in addition to the main BC score.

Exam format for both AB and BC

Section	Part	Time	Weight	Notes
I	A	60 min	50% total	30 multiple-choice questions; calculator not permitted
I	B	45 min		15 multiple-choice questions; graphing calculator required
II	A	30 min	50% total	2 free-response questions; graphing calculator required
II	B	60 min		4 free-response questions; calculator not permitted

11.2 The current testing format

As of the 2025–26 cycle, AP Calculus AB and BC are **hybrid digital** exams. Students answer multiple-choice questions in the Bluebook testing app and *view* the free-response questions there, but they handwrite their free-response answers in paper exam booklets.

That means you should prepare in two ways:

- practice reading, pacing, and selecting efficiently on-screen for multiple choice;
- practice writing organized, legible free-response work by hand.

In calculator-allowed parts, Bluebook provides a built-in Desmos graphing calculator, and approved handheld graphing calculators are also allowed under College Board policy.

11.3 How AP points are really won

A strong AP Calculus response usually does four things:

1. writes the correct setup,
2. uses correct notation,
3. shows enough reasoning for the reader to award the method points,
4. and answers the actual question in context.

What readers like to see

- Definite integrals written with limits and differentials.
- Derivatives written with respect to the correct variable.
- Units on contextual rates.
- A sentence of interpretation when a problem asks what a number means.
- Work arranged so the important step is easy to find.

What routinely loses points

- A bare final answer with no setup on a method-scored free-response part.
- Solving for a number when the problem asked for a rate or interpretation.
- Forgetting the constant of integration on indefinite integrals.
- Mixing up displacement and total distance.
- Using a calculator when a part is explicitly no-calculator.

11.4 Time strategy

Do not spend the first half of the exam chasing one stubborn question. The AP exam is a point-collection exercise, not a pride contest.

A practical pace:

- **Multiple choice:** move quickly, mark anything sticky, and return if time remains.
- **Free response:** write the setup immediately. Even when you are not sure of the full path, a correct setup often earns credit.
- **Calculator parts:** use the calculator to support reasoning, not replace it. Label every numerical answer.

11.5 What to memorize versus what to recognize

You should memorize:

- core derivative and integral formulas,
- FTC ideas,
- the major convergence tests and common series if you are in BC,

- the meaning of velocity, acceleration, displacement, and total distance,
- the polar and parametric formulas in BC.

You should recognize:

- when substitution is the right move,
- when a theorem such as IVT or MVT applies,
- when a graph or table is giving you enough information without an explicit formula,
- when the problem is asking for interpretation rather than calculation.

Practice

1. Which parts of the AP Calculus exam allow a graphing calculator?
2. In one sentence, explain the difference between displacement and total distance traveled.
3. What extra score does a BC student receive in addition to the main BC score?
4. If a free-response part asks for the meaning of $f'(3) = 2.4$ in context, what should a strong answer include besides the number 2.4?

Chapter takeaways

- Knowing calculus is not enough; you also need to know how the exam is built.
- FRQ points often come from setup, notation, and interpretation, not just the final number.
- Use technology where allowed, but never let it replace the mathematical story.

Chapter 12

Review Plans and Mixed Practice

Final synthesis chapter for AB and BC

Learning goals

- Turn the course into a repeatable review plan.
- Practice switching quickly among major AP Calculus ideas.
- Build a final checklist for exam week.

12.1 A one-week plan

If the exam is close, stop pretending you have time for a perfect rebuild. Use a tight loop:

1. Day 1: Limits, continuity, and derivative rules.
2. Day 2: Applications of derivatives and theorem justification.
3. Day 3: Integrals, FTC, accumulation, and applications of integration.
4. Day 4: Differential equations and calculator-active problems.
5. Day 5: BC-only topics or, for AB, extra free-response practice.
6. Day 6: Full mixed set under timed conditions.
7. Day 7: Error log review only; do not cram new topics.

12.2 A one-month plan

If you have more time, use a repeating structure:

- **Week 1:** Foundations and differentiation.
- **Week 2:** Applications of derivatives and integrals.
- **Week 3:** Differential equations, applications of integration, and BC extensions.
- **Week 4:** Full mixed review, released-style free response, and correction of every miss.

The important part is not the calendar. The important part is the *error log*. Keep one list of mistakes you make repeatedly. Most students do not need more problems; they need more honest correction.

Final checklist

- I can write and interpret a derivative in context.
- I know the difference between net change and total distance.
- I can use FTC Part 1 and Part 2 without confusing them.
- I can justify IVT, EVT, Rolle's Theorem, and MVT when asked.
- I know my calculator and when it is allowed.
- If I am in BC, I can work with series, parametric motion, and polar area.

12.3 Mixed practice

Work these without looking up a chapter first. That is the point.

Mixed Practice Set

1. Evaluate $\lim_{x\to 1} \frac{x^2-1}{x-1}$.
2. Differentiate $y = \ln(x^2+1)$.
3. A particle has velocity $v(t) = t^2 - 4$. On what intervals of t is it moving to the right?
4. Find the absolute maximum and minimum of $f(x) = x^3 - 3x$ on $[-2, 2]$.
5. Evaluate $\int_0^1 (2x+3)\,dx$.
6. If $g(x) = \int_0^x \sin(t^2)\,dt$, what is $g'(x)$?

7. Solve $\frac{dy}{dx} = 3x^2$ with $y(0) = 4$.
8. Find the area between $y = x$ and $y = x^2$ on $[0, 1]$.

BC For $x = t^2$, $y = t^3$, find $\frac{dy}{dx}$.

BC Determine whether $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges or diverges.

BC Find the third-degree Maclaurin polynomial for $\sin x$.

9. In a sentence, explain what it means if $C'(5) = -2.1$ dollars per shirt.

How to use your score on the mixed set

If you missed mostly algebra, go back to Chapter 0 and slow down. If you missed setup questions, revisit the examples and copy the structure. If you missed BC-only items but the AB items were strong, your foundation is fine and your BC layer needs targeted work. Be specific. Vague review wastes time.

Chapter takeaways

- Review works when it is honest, targeted, and tied to your actual mistakes.
- Mixed practice matters because the real exam does not announce the chapter for you.
- A clean error log is one of the fastest score improvers available.

Chapter A

Quick Reference Sheet

KEEP THIS CLOSE DURING REVIEW

A.1 Derivative formulas

Function	Derivative
c	0
x^n	nx^{n-1}
e^x	e^x
a^x	$a^x \ln a$
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$f(g(x))$	$f'(g(x))\, g'(x)$
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

A.2 Integral formulas

Integrand	Antiderivative
x^n $(n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$
e^x	$e^x + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$
$f'(x)[f(x)]^n$	$\frac{[f(x)]^{n+1}}{n+1} + C$ when $n \neq -1$
$\frac{f'(x)}{f(x)}$	$\ln\lvert f(x)\rvert + C$

A.3 Major AP formulas

- Tangent line at $x = a$:

$$y - f(a) = f'(a)(x - a)$$

- Average rate of change:

$$\frac{f(b) - f(a)}{b - a}$$

- FTC Part 2:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

- Average value:

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

- Area between curves:

$$\int_a^b (\text{top} - \text{bottom})\,dx$$

- Disk / washer:

$$V = \pi \int_a^b (R^2 - r^2)\,dx$$

- Separable differential equation:

$$g(y)\,dy = f(x)\,dx$$

- Euler's method:

$$y_{n+1} = y_n + f(x_n, y_n)\Delta x$$

- Parametric slope (BC):

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

- Polar area (BC):

$$A = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta$$

- Arc length of $y = f(x)$ (BC):

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

- Geometric series (BC):

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad |r| < 1$$

Chapter B

Current Unit Weightings and Course Map

Based on official College Board AB/BC course pages current in March 2026

B.1 AP Calculus AB unit weighting

Unit	MC Weight	Ch.
1. Limits and Continuity	10%–12%	1
2. Differentiation: Definition and Fundamental Properties	10%–12%	2
3. Composite, Implicit, and Inverse Functions	9%–13%	3
4. Contextual Applications of Differentiation	10%–15%	4
5. Analytical Applications of Differentiation	15%–18%	5
6. Integration and Accumulation of Change	17%–20%	6
7. Differential Equations	6%–12%	7
8. Applications of Integration	10%–15%	8

AB emphasis

The heaviest AB units are Unit 5 (analytical applications of differentiation) and Unit 6 (integration and accumulation of change). If your review time is limited, those are high-leverage places to get strong fast.

B.2 AP Calculus BC unit weighting

Unit	MC Weight	Ch.
1. Limits and Continuity	4%–7%	1
2. Differentiation: Definition and Fundamental Properties	4%–7%	2
3. Composite, Implicit, and Inverse Functions	4%–7%	3
4. Contextual Applications of Differentiation	6%–9%	4
5. Analytical Applications of Differentiation	8%–11%	5
6. Integration and Accumulation of Change	17%–20%	6
7. Differential Equations	6%–9%	7
8. Applications of Integration	6%–9%	8
9. Parametric Equations, Polar Coordinates, and Vector-Valued Functions	11%–12%	9
10. Infinite Sequences and Series	17%–18%	10

BC emphasis

In BC, the biggest weights sit in Unit 6 and Unit 10. That means FTC/integration fluency and series fluency matter a lot. BC also adds a Calculus AB subscore on top of the main BC score.

B.3 Exam-day reminders

- Both AB and BC have the same section timing and calculator split.
- As of the 2025–26 cycle, both exams are hybrid digital.
- BC includes a Calculus AB subscore in addition to the main BC score.

Chapter C

Selected Answers

Use to check your work, not to replace it

These are concise answer sketches, not full solutions. If you can match the final answer but cannot explain the setup, you are not done yet.

Chapter 0: Algebra and Function Toolkit

1. $(f \circ g)(2) = 2(2^2 - 3) + 1 = 3$ and $(g \circ f)(2) = (2 \cdot 2 + 1)^2 - 3 = 22$.
2. Write $y = \dfrac{x-4}{3}$, swap, and solve: $x = \dfrac{y-4}{3} \Rightarrow y = 3x + 4$. So $f^{-1}(x) = 3x + 4$.
3. $\ln(e^{3x}) + \ln 5 = 3x + \ln 5$.
4. $x = \dfrac{\pi}{6}$ and $x = \dfrac{5\pi}{6}$.
5. $\dfrac{f(3) - f(1)}{3 - 1} = \dfrac{10 - 2}{2} = 4$.
6. $\dfrac{1680 - 1200}{4} = 120$ people per year.

Chapter 1: Limits and Continuity

1. Factor: $\dfrac{(x-3)(x+3)}{x-3} = x + 3$, so the limit is 6.
2. Multiply by the conjugate to get $\dfrac{1}{\sqrt{x+4} + 2}$, so the limit is $\dfrac{1}{4}$.
3. No. The left-hand limit is -1 and the right-hand limit is 1.

4. $\lim_{x\to 2^-} \dfrac{1}{x-2} = -\infty$ and $\lim_{x\to 2^+} \dfrac{1}{x-2} = +\infty$.

5. No. The left-hand and right-hand limits are both 3, but $f(2) = 5$, so the function is not continuous at 2.

6. Because $-x^2 \le x^2 \sin(1/x) \le x^2$ and both bounding functions go to 0, the limit is 0.

7. There exists at least one c in $(1, 4)$ such that $f(c) = 0$; the function takes every value between -2 and 5 somewhere on the interval.

Chapter 2: Differentiation: Definition and Fundamental Properties

1. $f'(x) = \lim_{h\to 0} \dfrac{3(x+h)+1-(3x+1)}{h} = \lim_{h\to 0} \dfrac{3h}{h} = 3$.

2. $f'(x) = 20x^3 - 6x$.

3. $y' = 3x^2e^x + x^3e^x = e^x(3x^2 + x^3)$.

4. $y' = \dfrac{2x(x-1)-(x^2+1)}{(x-1)^2} = \dfrac{x^2-2x-1}{(x-1)^2}$.

5. $f'(x) = 2x$, so the slope at $x = 2$ is 4 and the line is $y - 3 = 4(x - 2)$.

6. $v(t) = 3t^2 - 12t + 9$ and $a(t) = 6t - 12$.

7. $y' = \cos x + \dfrac{1}{x}$.

Chapter 3: Differentiation: Composite, Implicit, and Inverse Functions

1. $y' = 6(x^2 + 4)^5(2x) = 12x(x^2 + 4)^5$.

2. $y' = \dfrac{1}{2}(1 + 3x^2)^{-1/2}(6x) = \dfrac{3x}{\sqrt{1+3x^2}}$.

3. $2x + (x\,y' + y) + 2yy' = 0$, so $(x + 2y)y' = -(2x + y)$ and $y' = -\dfrac{2x+y}{x+2y}$.

4. $(f^{-1})'(2) = \dfrac{1}{f'(3)} = \dfrac{1}{5}$.

5. $y' = \dfrac{2}{\sqrt{1-4x^2}}$.

6. Take logs: $\ln y = 2\ln x + 3\ln(x-1)$, so $\frac{y'}{y} = \frac{2}{x} + \frac{3}{x-1}$. Therefore $y' = x^2(x-1)^3\left(\frac{2}{x} + \frac{3}{x-1}\right)$.

7. $y' = x^x(\ln x + 1)$.

Chapter 4: Contextual Applications of Differentiation

1. Liters per minute.

2. $v(t) = 2t - 8$ and $a(t) = 2$.

3. A central-difference estimate is $\frac{4.7 - 4.1}{3.1 - 2.9} = 3$.

4. $A = \pi r^2$, so $\frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 2\pi(5)(3) = 30\pi \text{ cm}^2/\text{s}$.

5. For $f(x) = \sqrt{x}$, $f(9) = 3$ and $f'(9) = \frac{1}{6}$, so $L(9.2) = 3 + \frac{1}{6}(0.2) \approx 3.0333$.

6. $V = x^3$, so $dV = 3x^2\,dx = 3(4)^2(0.02) = 0.96 \text{ cm}^3$.

Chapter 5: Analytical Applications of Differentiation

1. $f'(x) = 3x^2 - 12 = 3(x^2 - 4)$, so the critical points are $x = \pm 2$.

2. $f'(x) = 3x^2 - 6x = 3x(x-2)$. Increasing on $(-\infty, 0)$ and $(2, \infty)$; decreasing on $(0, 2)$.

3. $f'(x) = 4x^3 - 8x = 4x(x^2 - 2)$ gives critical points $x = 0, \pm\sqrt{2}$. $x = 0$ is a local maximum and $x = \pm\sqrt{2}$ are local minima.

4. Yes. f is continuous on $[1, 5]$ and differentiable on $(1, 5)$. The average slope is $\frac{25-1}{4} = 6$, so $2c = 6$ and $c = 3$.

5. $f''(x) = 6x - 12$. Concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$.

6. Let sides be x and $48/x$. Then $P(x) = 2x + 96/x$. Differentiate: $P'(x) = 2 - 96/x^2 = 0$ gives $x^2 = 48$, so the rectangle is a square with side $4\sqrt{3}$.

7. Critical point at $x = 2$. Evaluate $f(-1) = 6$, $f(2) = -3$, $f(4) = 1$. Absolute maximum is 6 at $x = -1$; absolute minimum is -3 at $x = 2$.

Chapter 6: Integration and Accumulation of Change

1. $2x^3 - 2x^2 + x + C$.
2. $\left[\frac{3}{2}x^2 - x\right]_0^2 = 6 - 2 = 4$.
3. $g'(x) = \cos(x^2)$, so $g'(2) = \cos 4$.
4. Let $u = x^2 + 1$ so $du = 2x\,dx$. Then the integral is $2\int u^3\,du = \dfrac{1}{2}(x^2+1)^4 + C$.
5. With nodes $0, 1, 2$, $T_2 = \dfrac{1}{2}[f(0) + 2f(1) + f(2)] = \dfrac{1}{2}[0 + 2(1) + 4] = 3$.
6. By parts with $u = x$, $dv = \cos x\,dx$: $\int x\cos x\,dx = x\sin x + \cos x + C$.
7. Write $\dfrac{3}{x(x+3)} = \dfrac{1}{x} - \dfrac{1}{x+3}$, so the integral is $\ln|x| - \ln|x+3| + C$.
8. It diverges. Writing it as $\lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\ln b$, the limit is infinite.

Chapter 7: Differential Equations

1. The population's rate of change is 30% of its current size; the bigger P is, the faster it grows.
2. Integrate: $y = 2x^2 + C$. Use $y(1) = 2$ to get $2 = 2 + C$, so $C = 0$ and $y = 2x^2$.
3. The solution is $y = 5e^x$.
4. $P = 0$ and $P = 100$.
5. At $(0, 2)$ the slope is $0 - 2 = -2$, so $y_1 = 2 + (-2)(0.1) = 1.8$.
6. Since $y - 4 = 1 - 4 = -3 < 0$, the solution is decreasing there.

Chapter 8: Applications of Integration

1. Displacement: $\int_0^5 (t-2)\,dt$. Total distance: $\int_0^5 |t-2|\,dt$.
2. Intersections satisfy $2x = x^2$, so $x = 0, 2$. Area $= \int_0^2 (2x - x^2)\,dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 = \frac{4}{3}$.
3. $\dfrac{1}{2}\int_0^2 x^2\,dx = \dfrac{1}{2}\cdot\dfrac{8}{3} = \dfrac{4}{3}$.
4. $V = \pi\int_0^2 x^2\,dx = \pi\left[\dfrac{x^3}{3}\right]_0^2 = \dfrac{8\pi}{3}$.

5. Cross-sectional area is $A(x) = x^2$, so $V = \int_0^3 x^2\,dx = 9$.

6. $L = \int_1^3 \sqrt{1 + \left(\frac{1}{x}\right)^2}\,dx$.

Chapter 9: Parametric Equations, Polar Coordinates, and Vector-Valued Functions

1. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{2t}$ (for $t \neq 0$).

2. First $\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2}$. Then $\frac{d^2y}{dx^2} = \frac{d}{dt}(3t/2)\big/(2t) = \frac{3/2}{2t} = \frac{3}{4t}$.

3. $\mathbf{v}(t) = \langle 3, 2t \rangle$ and speed $= \sqrt{9 + 4t^2}$.

4. $L = \int_0^1 \sqrt{(1)^2 + (2t)^2}\,dt = \int_0^1 \sqrt{1 + 4t^2}\,dt$.

5. $A = \frac{1}{2}\int_0^{2\pi} (1 + \cos\theta)^2\,d\theta$.

6. Since $r = 2\sin\theta$, multiply by r: $r^2 = 2r\sin\theta \Rightarrow x^2 + y^2 = 2y$, a circle centered at $(0, 1)$ with radius 1.

Chapter 10: Infinite Sequences and Series

1. Yes. Divide top and bottom by n: $\frac{3 + 1/n}{1 + 4/n} \longrightarrow 3$.

2. It is geometric with ratio $\frac{1}{4}$, so it converges to $\frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$.

3. It diverges; this is the harmonic series.

4. It converges; this is a p-series with $p = 3 > 1$.

5. It converges absolutely because $\sum \frac{1}{n^2}$ converges.

6. The ratio test gives $\left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \frac{|x|}{n+1} \longrightarrow 0$, so the series converges for every real x.

7. $T_2(x) = 1 - \frac{x^2}{2}$.

8. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$.

Chapter 11: Exam Structure, Scoring Habits, and Strategy

1. Multiple Choice Part B and Free Response Part A.
2. Displacement is the net change in position, while total distance traveled counts all motion regardless of direction.
3. A Calculus AB subscore.
4. It should state that at the input or time $x = 3$, the quantity is changing at a rate of 2.4 output-units per input-unit, with the correct units and contextual meaning.

Chapter 12: Review Plans and Mixed Practice

1. 2.
2. $y' = \dfrac{2x}{x^2 + 1}$.
3. The particle moves right when $v(t) > 0$, so when $t^2 - 4 > 0$, i.e. $t < -2$ or $t > 2$.
4. Critical points at $x = \pm 1$. Checking $x = -2, -1, 1, 2$ gives values $-2, 2, -2, 2$, so the absolute maximum is 2 at $x = -1$ and $x = 2$, and the absolute minimum is -2 at $x = -2$ and $x = 1$.
5. $\left[x^2 + 3x\right]_0^1 = 4$.
6. $g'(x) = \sin(x^2)$.
7. Integrate: $y = x^3 + C$. Since $y(0) = 4$, $C = 4$, so $y = x^3 + 4$.
8. $\int_0^1 (x - x^2)\,dx = \frac{1}{6}$.
9. $\dfrac{dy}{dx} = \dfrac{3t^2}{2t} = \dfrac{3t}{2}$ for $t \neq 0$.
10. It converges; it is a p-series with $p = 2 > 1$.
11. $T_3(x) = x - \dfrac{x^3}{6}$.
12. At a production level of 5 shirts, the cost is decreasing at a rate of \$2.10 per additional shirt (with the correct contextual interpretation).

Chapter D

Sources for AP Alignment

Course structure and exam-format references

The calculus explanations, examples, and practice in this book are original. The AP-specific course structure and exam-format notes were aligned against official College Board pages current as of March 2026, including the following:

- AP Calculus AB course page (unit structure and multiple-choice weighting)
- AP Calculus BC course page (unit structure and multiple-choice weighting)
- AP Calculus AB exam page (section timing, calculator split, hybrid digital overview)
- AP Calculus BC exam page (section timing, calculator split, hybrid digital overview)
- Hybrid Digital AP Exams overview page
- AP Exam terms and conditions / calculator guidance for Bluebook calculator availability
- Special Score Structure: Calculus BC page for the AB subscore note

Because College Board can update policies and testing details, always verify logistics with the current official pages before exam day.